CSE 532
Assignment 2
Due Thursday, February 5, 1997

1. In this problem we consider the power of an alternating pushdown automaton. We know that the one-way[1] nondeterministic pushdown automata accept exactly the context-free languages. Define APDA as the class of languages accepted by one-way alternating pushdown automata. There is no way we can force these automata to halt, so we don't make that a requirement. As it happens

$$\mathrm{APDA} = \mathrm{EXPTIME} = \bigcup_{c>0} \mathrm{TIME}(2^{cn}).$$

   (a) Prove that EXPTIME $\subseteq$ APDA. (Hint: it would be best to use the alternating space theorem and show that ASPACE$(n) \subseteq$ APDA. Clearly, your APDA will have to do a search through an and-or tree, but because of the alternation, it has the luxury of doing some of the searching in parallel.)

   (b) (Extra Credit) Prove that APDA $\subseteq$ EXPTIME. (This is somewhat difficult, but I can give concrete hints to anyone interested in doing this.)

2. The polynomial hierarchy, Σ_i^P, $1 \leq i$, has a number of equivalent definitions. Perhaps the simplest is using quantifiers. A language L is a member of Σ_i^P if and only if

$$L = \{x : \exists x_1 \forall x_2 \cdots Q_i x_i (R(x, x_1, x_2, \ldots, x_i) \text{ and for all } i, |x_i| \leq |x|^m)\},$$

   where R is a polynomial time computable relation, m is a constant, and $Q_i = \forall$ for i even and $Q_i = \exists$ for i odd. Note that $\Sigma_1^P = \mathrm{NP}$.

   (a) Show that $\Sigma_i^P \subseteq \Sigma_{i+1}^P$ for all $i \geq 1$.

   (b) Show that $L \in \Sigma_i^P$ if and only if there is an alternating Turing machine that runs in polynomial time, starts in an existential state and makes at most $i-1$ alternations before terminating. That is, the machine runs existentially for a while, the moves universally for a while, and so on for at most $i-1$ changes in alternation, then finally terminates in an accepting or rejecting state.

[1] One-way means that the input head cannot move to the left, although it can remain stationary or move right

(c) Show that $\Sigma_i^P \subseteq \text{PSPACE}$ for all $i \geq 1$.

(d) Show that if for some i, $\Sigma_i^P \neq \Sigma_{i+1}^P$, then NP $\neq$ PSPACE.

(e) (Extra Credit) Define B_i to be the set of valid quantified boolean formulas with at most $i-1$ alternations of quantifiers, starting with existential quantifiers. For example the following formula is in B_2.

$$\exists x \exists y \forall z (x \equiv z \vee y \equiv z).$$

Prove that B_i is polynomial time complete for Σ_i^P. That is, $B_i \in \Sigma_i^P$ and for all $L \in \Sigma_i^P$, L is polynomial time reducible to B_i.

3. We can define the alternating linear space hierarchy, Σ_i^{LIN}, $1 \leq i$, as follows. The language L is a member of Σ_i^{LIN} if and only if there is an alternating linear space bounded Turing machine that starts in an existential state and makes at most $i-1$ changes in alternation before terminating. Prove that this hierarchy is no hierarchy at all, that is:

$$\Sigma_i^{LIN} = \Sigma_{i+1}^{LIN},$$

for all $i \geq 1$. (Note that $\Sigma_1^{LIN} = \text{NSPACE}(n)$.)