CSE 532 Spring 2008
Computational Complexity II
Problem Set #1
Due: May 2, 2008

Problems:

1. Show that if $\mathsf{NP} \subseteq \mathsf{TIME}(n^{\log n})$ then $\mathsf{PH} \subseteq \bigcup_k \mathsf{TIME}(n^{\log^k n})$. $\Sigma_k^p \subseteq \mathsf{TIME}(n^{\log^{ck} n})$.

2. Show that $\#2\text{-}SAT$ is #P-complete.

3. This problem will derive Lupanov's bound on the worst-case size required to compute any Boolean function on n bits. The key to this construction is to compute many functions of fewer than n bits using a single circuit that has more than one node as an output node.

   (a) Show how to compute all conjunctions from $\{x_1, \ldots, x_m\}$ efficiently using a single circuit.

   (b) View the inputs to f as defining a $2^k \times 2^{n-k}$ matrix. For some parameter $s \leq 2^k$ partition the rows of f into groups of size s and one remaining group. Then represent f as $\bigvee_{i,v}(f_{i,v}(x_1, \ldots, x_k) \wedge f'_{i,v}(x_{k+1}, \ldots, x_n))$ for $i \in [p]$ and $v \in \{0,1\}^s$ where $p = \lceil 2^k/s \rceil$, each of the functions $f_{i,v}$ is 1 on at most s inputs, and for each $i \in [p]$ the functions $f'_{i,v}$ taken together are 1 on at most 2^{n-k} inputs.

   (c) Use the properties of part (b) to find an efficient construction using the circuit from (a) that computes f. Then choose values of k and s to optimize the construction and derive a size $2^n/n + o(2^n/n)$ circuit that computes f.

4. Even if $\mathsf{P} = \mathsf{NP}$ we do not know whether $\#\mathsf{P} \subseteq \mathsf{FP}$.

   (a) Show that if $\mathsf{P} = \mathsf{NP}$ then for every $f \in \#\mathsf{P}$ there is a randomized algorithm that approximates f within a factor of 2. Hint: Use a hashing-based method for estimating the size of a set, as given in the proof of the Valiant-Vazirani lemma or Lautemann's Lemma.

   (b) Under the same assumption, improve the approximation factor.