CSE 531: Computability and Complexity — Autumn 2005
Problem Set #3 (On P and NP) — Instructor: Paul Beame
Due on Thursday, **November 10, 2005** in class.

Instructions: Same as Problem Set 1.

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1. (10 points) Assuming $\mathsf{P} = \mathsf{NP}$, describe a polynomial time algorithm that, given a Boolean formula ϕ, actually produces a satisfying assignment for ϕ if it is satisfiable. (Hint: Use a satisfiability tester repeatedly to find the assignment bit-by-bit.)

2. (10 points) Let $2SAT = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable 2CNF formula}\}$. Show that $2SAT \in \mathsf{P}$.

3. (10 points) Define $M2SAT = \{\langle\phi, k\rangle \mid \phi$ is a 2CNF formula for which there is an assignment satisfying at least k of its clauses$\}$. Prove that $M2SAT$ is NP-complete.

4. (10 points) Sipser's text 2nd edition problem 7.30: This problem is inspired by the single-player game *Minesweeper*, generalized to an arbitrary graph. Let G be an undirected graph, where each node either contains a single, hidden *mine* or is empty. The player chooses nodes, one by one. If the player chooses a node containing a mine, the player loses. If the player chooses an empty node, the player learns the number of neighboring nodes containing mines. (A neighboring node is one connected to the chosen node by an edge.) The player wins if and when all empry nodes have been so chosen.

   In the *mine consistency problem* you are given a graph G, along with numbers labeling some of G's nodes. You must determine whether a placement of mines on the remaining nodes is possible, so that any node v that is labeled m has exactly m neighboring nodes containing mines. Formulate this problem as a language and show that it is NP-complete.

5. (10 points) Say that two Boolean formulas are *equivalent* if they have the same set of variables and are true on the same sets of assignments to those variables (i.e. they describe the same Boolean function). A Boolean formula is *minimal* if no shorter Boolean formula is equivalent to it. let MIN-FORMULA be the set of minimal Boolean formulas. Show that if $\mathsf{P} = \mathsf{NP}$ then MIN-FORMULA$\in \mathsf{P}$.

6. (10 points) Recall that you may consider circuits that output strings over $\{0,1\}$ by designating several gates as output gates. Let $add_n : \{0,1\}^{2n} \to \{0,1\}^{n+1}$ take the sum of two n bit binary integers and produce the $n+1$ bit result. Show that you can compute the add_n function with circuits that are simultaneously $O(n)$ size and $O(\log n)$ depth.

7. (Extra credit) A bipartite graph is a graph whose vertices can be partitioned into two disjoint parts each of which is an independent set. Formally a bipartite graph $H = (X, Y, E)$ has vertex set $X \cup Y$ for disjoint sets X, Y and each edge in its edge set E has one endpoint in X and one in Y. A k-bipartite clique of H is a pair of subsets $S \subseteq X$ and $T \subseteq Y$ with $|S| = |T| = k$ such that $(s,t) \in E$ for each $s \in S$ and $t \in T$ (informally all "cross-edges" exist between S and T). Define the language

$$\text{BIPARTITE-CLIQUE} = \{\langle H, k\rangle \mid H \text{ is a bipartite graph that has a } k\text{-bipartite clique}\}\ .$$

Prove that BIPARTITE-CLIQUE is NP-complete.
(Hint: This problem is reasonably tricky and the obvious reduction from CLIQUE that one would be tempted to try does *not* work.)