CSE 531: Computability and Complexity — Autumn 2005
Problem Set #2 — Instructor: Paul Beame
Due on Tuesday, **November 1, 2005** in class.

Instructions: Same as Problem Set 1.

---

1. (15 points) In class, we have discussed various undecidability results relating to Turing machines such as Rice's Theorem that deal with the nature of language they recognize. In this problem, we consider questions concerning the actual runtime behavior of Turing machines. Which of the following problems about Turing machines are decidable and which are not? Briefly justify your answers.
   (a) To determine, given a Turing machine M, a string w, and integers a and b does M run for more than $a|w|^2 + b$ steps on input w?
   (b) To determine, given a Turing machine M, and integers a and b does there exist a string w such that M runs for more than $a|w|^2 + b$ steps on input w?
   (c) To determine, given a Turing machine M does M have a *useless state*, a state that is never entered on on any input string.

2. (10 points) Define
$$\text{REGULAR}_{\text{CFG}} = \{\langle G\rangle \mid G \text{ is a context-free grammar and } L(G) \text{ is regular}\}.$$
   Prove that $\text{REGULAR}_{\text{CFG}}$ is undecidable. Ideas similar to those used to prove that ALL_{CFG} is undecidable may be useful.

3. (10 points) Sipser's text, 1st Edition Problem 5.19 (equivalently 2nd Edition Problem 5.21).

4. (10 points) A language L is said to be $\leq_m$*-complete* for a class of languages $\mathcal{C}$ if and only if
   - $L \in \mathcal{C}$ and
   - For all languages $K \in \mathcal{C}$, $K \leq_m L$.

   Prove that A_{TM} and $HALT_{\text{TM}}$ are both $\leq_m$-complete for the class of Turing-recognizable languages.

5. (15 points) We say that a relation $R \subseteq (\{0,1\}^*)^k$ is decidable iff the language
$$L_R = \{\langle x_1, x_2, \ldots, x_k\rangle \mid (x_1, x_2, \ldots, x_k) \in R\}$$
   is decidable.

   Define Σ_k, for $k \geq 0$, to be the class of all languages L for which there is a decidable $(k+1)$-ary relation R such that
$$L = \{x \mid \exists x_1 \forall x_2 \cdots Q_k x_k \; R(x_1, x_2, \ldots, x_k, x)\},$$

where the quantifier Q_k is $\exists$ if k is odd and $\forall$ if k is even.
Define $\Pi_k = \{\overline{L} \mid L \in \Sigma_k$, i.e. Π_k is the set of all complements of languages in Σ_k.
Equivalently, Π_k is the class of all languages L for which there is a decidable $(k+1)$-ary relation R such that

$$L = \{x \mid \forall x_1 \exists x_2 \cdots Q_k x_k \; R(x_1, x_2, \ldots, x_k, x)\} \ ,$$

where the quantifier Q_k is $\forall$ if k is odd and $\exists$ if k is even.

In this notation, clearly Σ_0 and Π_0 equal the set of decidable languages. Moreover, by problem 2 on homework 1, Σ_1 consists of precisely the Turing-recognizable languages.

(a) Prove that the language

$$\mathrm{E}_{\mathrm{TM}} = \{\langle M \rangle \mid M \text{ is a Turing machine and } L(M) = \emptyset\}$$

is $\leq_m$-complete for Π_1.

(b) Prove that the language

$$\mathrm{ALL}_{\mathrm{TM}} = \{\langle M \rangle \mid M \text{ is a Turing machine and } L(M) = \{0,1\}^*\}$$

is in Π_2 but not in Σ_1 or Π_1.

(c) Find the smallest k you can such that the language

$$\mathrm{EQ}_{\mathrm{TM}} = \{\langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are Turing machines and } L(M_1) = L(M_2)\}$$

is in Σ_k or Π_k and justify your answer.

(d) (Extra credit) Prove that $\mathrm{EQ}_{\mathrm{TM}}$ is $\leq_m$-complete for the class you found in part (c).