CSE 531: Computability and Complexity
Autumn 2004
SAMPLE FINAL EXAM

Instructions: Attempt all questions. The exam is for a maximum of 160 points. It has **six** questions, and you have 1 hour and 50 minutes to answer them. You may use without proof any of the theorems we have proved in class or which are proved in the textbook. This is an open book/notes exam, but reference to any "outside" sources is not allowed.

---

1. For each of the following assertions, state whether they are True, False, or Open according to our current state of knowledge of computability and complexity theory, as described in class. You do *not* have to justify your answer choice. ($3 \times 10 = 30$ points)
   (a) $A_{\text{TM}} \leq_m TQBF$
   (b) EXPSPACE contains all decidable languages.
   (c) NL $\neq$ PSPACE
   (d) P $=$ NP $\cap$ coNP
   (e) coNP $=$ NEXPTIME
   (f) BPP $=$ PSPACE
   (g) $HAMPATH \in$ coNP
   (h) $2SAT \leq_P CLIQUE$
   (i) $CLIQUE \leq_P 2SAT$
   (j) P contains all context-free languages.

2. (a) (20 points) Define the language $SELFACCEPT = \{\langle M\rangle \mid M$ is a Turing machine and $\langle M\rangle \in L(M)\}$. Prove that $SELFACCEPT$ is undecidable.
   (b) (15 points) Let S be an infinite, Turing-recognizable language. Prove that S has an infinite, decidable subset.

3. (25 points) Let $EQ_{\text{BP}} = \{\langle B_1, B_2\rangle \mid B_1$ and B_2 are branching programs that compute the same Boolean function$\}$. (This is similar to the language we studied in class but with the read-once restriction on the branching programs removed.) Prove that EQ_{BP} is coNP-complete.

4. (25 points) A directed cycle in a directed graph $G = (V, E)$ is a sequence of k distinct nodes $v_1, \ldots, v_k \in V$ for some $k \geq 2$ where there is a directed edge from v_i to v_{i+1} for $1 \leq i < k$ and from v_k to v_1. Define the language

   $DAG = \{\langle G\rangle \mid G$ is a directed acyclic graph, i.e. a directed graph that has **no** directed cycle$\}$.

   Prove that DAG is NL-complete.

5. (20 points) In this problem, we let M be a deterministic Turing machine, w a string, i and j binary integers, and $\alpha \in Q \cup \Gamma$ where Q is the set of states of M and Γ is its tape alphabet.

   Let $A = \{\langle M, w, i, j, \alpha\rangle \mid \alpha$ is the i'th symbol of the configuration after the j'th step of the computation of M on input $w\}$.

   Prove that A is EXPTIME-complete (under polynomial time reductions).

6. (25 points)

(a) Prove that if $L \in \mathrm{BPP}$, then there is a polynomially bounded function $p : \mathbb{N} \to \mathbb{N}$ and a polynomial time *deterministic* Turing machine V such that

$$
\begin{aligned}
x \in L &\implies \mathrm{Prob}_{r \in \{0,1\}^{p(|x|)}}[V \text{ accepts } (x,r)] > (1 - 2^{-|x|}) \\
x \notin L &\implies \mathrm{Prob}_{r \in \{0,1\}^{p(|x|)}}[V \text{ accepts } (x,r)] < 2^{-|x|} \ ,
\end{aligned}
$$

where the probabilities are taken over r chosen uniformly at random from $\{0,1\}^{p(|x|)}$ and $|x|$ denotes the length of the string x.

(b) Let $n \geq 1$, and V be the TM from Part (a). For each $x \in \{0,1\}^n$, let $\mathrm{BAD}_x = \{r \in \{0,1\}^{p(n)} \mid V$ gives the wrong answer on (x,r), i.e., V rejects (x,r) if $x \in L$ or V accepts (x,r) if $x \notin L\}$. Prove that there exists $r \in \{0,1\}^{p(n)}$ such that $r \notin \bigcup_{x \in \{0,1\}^n} \mathrm{BAD}_x$.

(c) Using Part (b) above, argue that every language in BPP has a circuit family of polynomial size that decides it.