CSE 531: Computability and Complexity — Autumn 2002
Problem Set #3 — Instructor: Venkatesan Guruswami
Due on Tuesday, **November 12, 2002** in class.

Instructions: You are permitted (though not exactly encouraged) to collaborate with fellow students taking the class in solving problem sets. If you do so, *please indicate for each problem the people you worked with on that problem.* Note that you must write down solutions on your own and collaboration must be restricted to a discussion of solution ideas. Collaboration is **not allowed** for the optional problem. Solutions are expected to be your original work and so you must refrain from looking up solutions or solution ideas from websites or other literature.

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1. Problem 7.12, Sipser's book ($MODEXP$ is in P)

2. Problem 7.30, Sipser's book (MAX-CLIQUE is in P if P = NP)

3. Problem 7.26, Sipser's book ($PUZZLE$ is NP-complete)

4. (a) Problem 7.22, Sipser's book. ($\neq SAT$ is NP-complete)

   (b) A graph $G = (V, E)$ is said to be k-colorable if there exists a map $f : V \to \{1, 2, \ldots, k\}$ such that $f(u) \neq f(v)$ whenever $(u, v) \in E$ (i.e. endpoints of every edge get distinct colors under the map f). Define the language

   $$3COLOR = \{\langle G\rangle \mid G \text{ is 3-colorable}\} .$$

   Prove that $\neq SAT$ (defined in Part (a) above) is polynomial time mapping reducible to $3COLOR$. Conclude that $3COLOR$ is NP-complete.

5. Define MAJ-3SAT to be the language of 3-CNF formulas ϕ for which there is an assignment to the variables which satisfies *at least two* out of the three literals in *every* clause. For example, $(x \vee y \vee \bar{z}) \wedge (\bar{x} \vee \bar{y} \vee \bar{z}) \wedge (y \vee z \vee w)$ is in MAJ-3SAT since the assignment $x = 0$, $y = 1$, $z = 0$ and $w = 1$ satisfies at least two literals in each clause. Prove that MAJ-3SAT is in P.

6. Define the Boolean function $\text{MAJORITY}_n : \{0, 1\}^n \to \{0, 1\}$ as:

   $$\text{MAJORITY}_n(x_1, x_2, \ldots, x_n) = 1 \text{ if and only if } \sum_{i=1}^{n} x_i \geq n/2 .$$

   Thus MAJORITY_n returns the majority vote of its inputs. Show that MAJORITY_n can be computed with $O(n)$ size Boolean circuits (with NOT gates and fan-in 2 AND and OR gates). (Hint: Divide and Conquer)

7. * **(Optional Problem)** A bipartite graph is a graph whose vertices can be partitioned into two disjoint parts each of which is an independent set. Formally a bipartite graph $H = (X, Y, E)$ has vertex set $X \cup Y$ for disjoint sets X, Y and each edge in its edge set E has one endpoint in X and one in Y. A k-bipartite clique of H is a pair of subsets $S \subseteq X$ and $T \subseteq Y$ with $|S| = |T| = k$ such that $(s, t) \in E$ for each $s \in S$ and $t \in T$ (informally all "cross-edges" exist between S and T). Define the language

   $$\text{BIPARTITE-CLIQUE} = \{\langle H, k\rangle \mid H \text{ is a bipartite graph that has a } k\text{-bipartite clique}\} .$$

   Prove that BIPARTITE-CLIQUE is NP-complete.