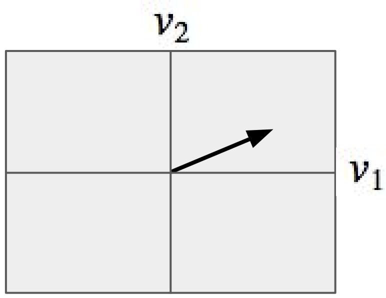
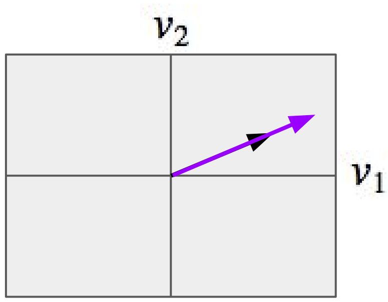
VECTORS

2-D

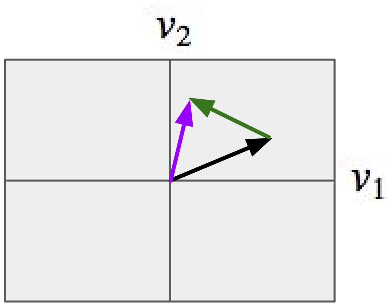
general description



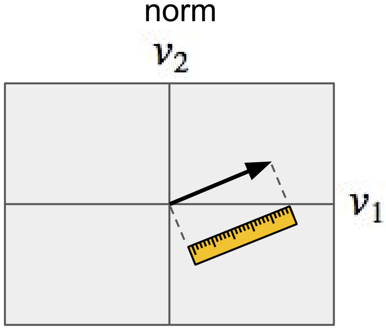
scalar multiplication



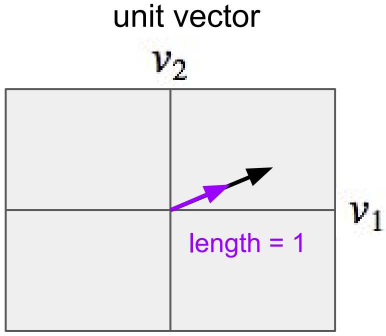
addition



norm



unit vector



N-D

general description

column representation

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$

row representation

$$\mathbf{v}^T = (v_1 \quad v_2 \quad \ldots \quad v_N)$$

scalar multiplication

$$c\mathbf{v} = c\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} = \begin{pmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_N \end{pmatrix}$$

addition

$$\mathbf{v} + \mathbf{u} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_N + u_N \end{pmatrix}$$

norm

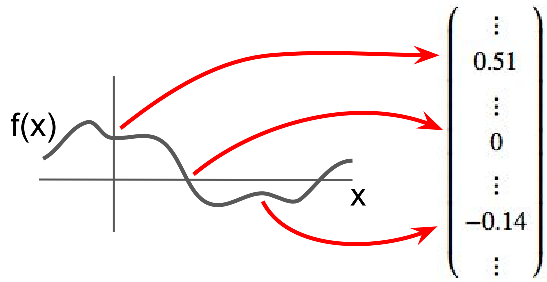
$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_N^2}$$

unit vector

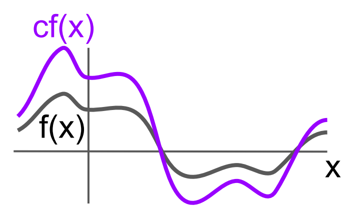
$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

∞-D

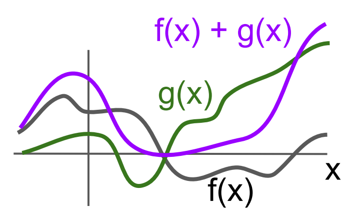
general description



scalar multiplication



addition

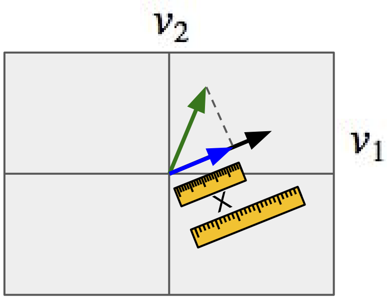
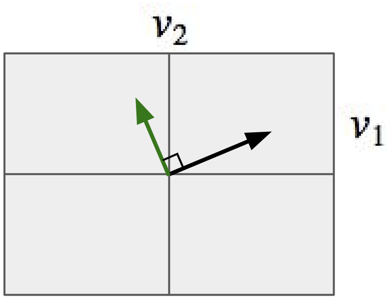
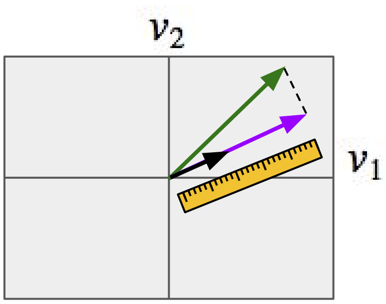
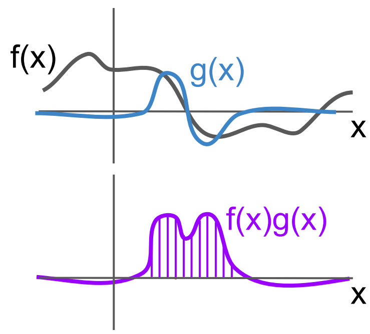


norm

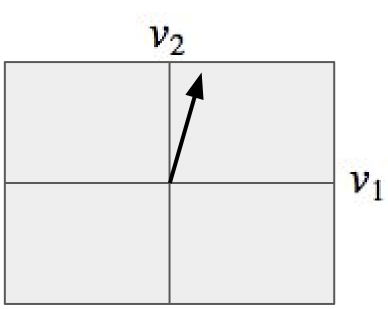
$$\|f\| = \sqrt{\int_{x_1}^{x_2} f(x)^2 dx}$$

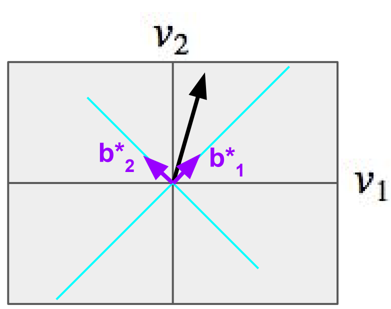
unit vector

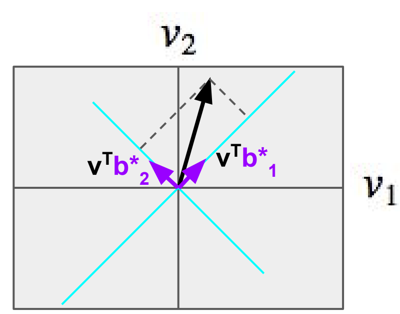
$$f^{unit} = \frac{f}{\|f\|}$$

2-D	N-D	∞-D
dot (inner) product 	dot (inner) product $\mathbf{v}\cdot\mathbf{u} = \ < \mathbf{v},\mathbf{u} > \ = \mathbf{v}^T\mathbf{u} =$ $v_1u_1 + v_2u_2 + \cdots + v_Nu_N$ takes in two vectors, returns scalar	dot (inner) product $< f, g >= \int_{x_1}^{x_2} f(x)g(x)dx$
orthogonality 	orthogonality two vectors are *defined* to be orthogonal if $\mathbf{v}\cdot\mathbf{u} = \ < \mathbf{v},\mathbf{u} > \ = \mathbf{v}^T\mathbf{u} = 0$	orthogonality two functions are *defined* to be orthogonal if $< f, g >= \int_{x_1}^{x_2} f(x)g(x)dx = 0$
projection (filtering) 	projection (filtering) $\hat{\mathbf{v}}\cdot\mathbf{u} = \ < \hat{\mathbf{v}},\mathbf{u} > \ = \hat{\mathbf{v}}^T\mathbf{u}$	projection (filtering)  feature detection $\int_{x_1}^{x_2} f(x)g(x)dx$ $\lVert g \rVert = 1$

Example: rewrite the following vector in a coordinate system rotated 45 degrees CCW







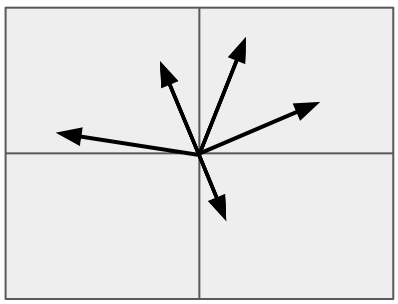
$$\mathbf{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\mathbf{b}^*{}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \mathbf{b}^*{}_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad \mathbf{v}^* = \begin{pmatrix} \mathbf{v}^T\mathbf{b}^*{}_1 \\ \mathbf{v}^T\mathbf{b}^*{}_2 \end{pmatrix}$$

$$\mathbf{v}^* = \begin{pmatrix} 1/\sqrt{2} + 4/\sqrt{2} \\ -1/\sqrt{2} + 4/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 5/\sqrt{2} \\ 3/\sqrt{2} \end{pmatrix} \approx \begin{pmatrix} 3.54 \\ 2.12 \end{pmatrix}$$

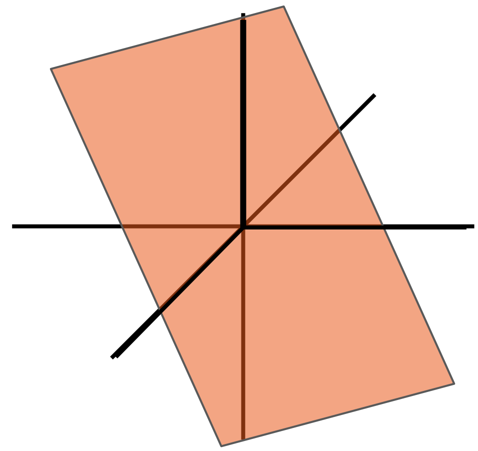
VECTOR SPACES

vectors live in a *vector space*



2D vector space

one vector space can be a *subspace* of another vector space



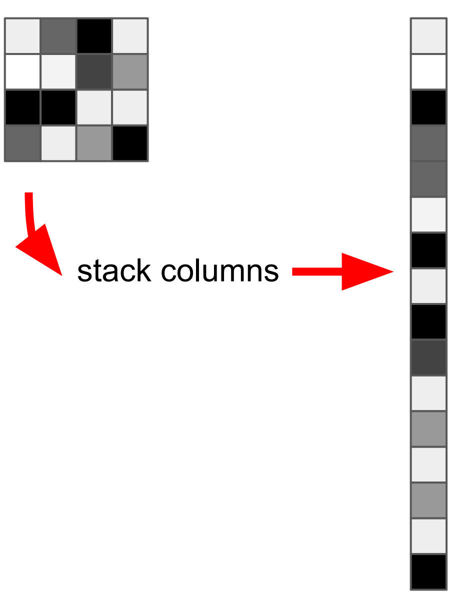
2D vector space as a subspace of a 3D vector space

TIPS AND TRICKS

Always ask:

- Is the quantity I'm working with a scalar? vector? function?
    - If vector: what is dimensionality? what does dimensionality represent? what do indices represent? what do elements represent?
    - If function: what is domain of function? what does domain of function represent? what does function argument represent? what do function values represent?

representing an image as a vector



inner product between functions of multiple arguments

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)g(x,y)\,dx\,dy$$

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y,t)g(x,y,t)\,dx\,dy\,dt$$

2 x 2	M x N

general description (2 x 2)

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

general description (M x N)

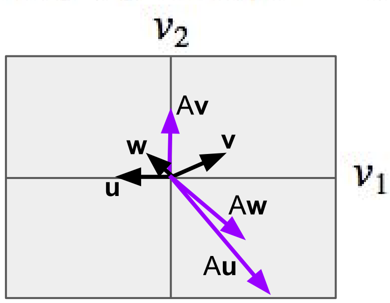
$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \ddots & a_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ a_{M1} & a_{N2} & \dots & a_{MN} \end{pmatrix}$$

$$= \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1^c & \mathbf{a}_2^c & \dots & \mathbf{a}_N^c \\ \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} \leftarrow & (\mathbf{a}_1^r)^T & \rightarrow \\ \leftarrow & (\mathbf{a}_2^r)^T & \rightarrow \\ \leftarrow & \vdots & \rightarrow \\ \leftarrow & (\mathbf{a}_M^r)^T & \rightarrow \end{pmatrix}$$

matrix times vector (2 x 2)

$$A\mathbf{v} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} =$$

$$\begin{pmatrix} [-1 \times 2] + [2 \times 1] \\ [2 \times 2] + [-1 \times 1] \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$



think of a matrix in terms of its action on a vector space

matrix times vector (M x N)

$$A\mathbf{v} = \begin{pmatrix} (\mathbf{a}_1^r)^T \mathbf{v} \\ (\mathbf{a}_2^r)^T \mathbf{v} \\ \vdots \\ (\mathbf{a}_M^r)^T \mathbf{v} \end{pmatrix} = v_1 \mathbf{a}_1^c + v_2 \mathbf{a}_2^c + \cdots + v_N \mathbf{a}_N^c$$

dot product of each row with **v**
or
weighted sum of columns,
weighted by elements of **v**

matrix times matrix (2 x 2)

$$AB = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} =$$

$$\begin{pmatrix} [-1 \times 2] + [2 \times 1], [-1 \times 3] + [2 \times -1] \\ [2 \times 2] + [-1 \times 1], [2 \times 3] + [-1 \times -1] \end{pmatrix} =$$

$$\begin{pmatrix} 0 & -5 \\ 3 & 7 \end{pmatrix}$$

$$AB \neq BA$$

matrix times matrix (M x N)

$$AB = \begin{pmatrix} (\mathbf{a}_1^r)^T \mathbf{b}_1^c & (\mathbf{a}_1^r)^T \mathbf{b}_2^c & \dots & (\mathbf{a}_1^r)^T \mathbf{b}_P^c \\ (\mathbf{a}_2^r)^T \mathbf{b}_1^c & (\mathbf{a}_2^r)^T \mathbf{b}_2^c & \dots & (\mathbf{a}_2^r)^T \mathbf{b}_P^c \\ \vdots & \ddots & & \\ (\mathbf{a}_M^r)^T \mathbf{b}_1^c & (\mathbf{a}_M^r)^T \mathbf{b}_2^c & \dots & (\mathbf{a}_M^r)^T \mathbf{b}_P^c \end{pmatrix} =$$

$$\begin{pmatrix} \uparrow & \uparrow & \dots & \uparrow \\ A\mathbf{b}_1^c & A\mathbf{b}_2^c & \dots & A\mathbf{b}_P^c \\ \downarrow & \downarrow & \dots & \downarrow \end{pmatrix}$$

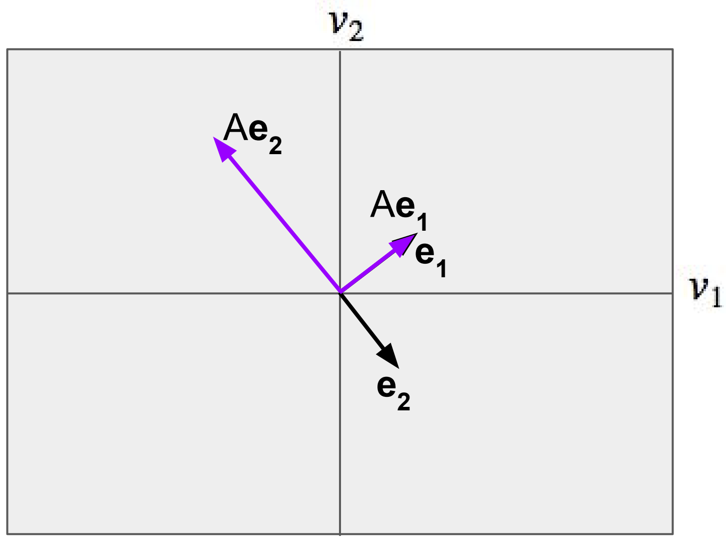
each column of result is *A* multiplied by corresponding column of B

EIGENVECTORS

CSE 528

2 x 2

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$



$$\lambda_1 = -3, \;\; \lambda_2 = 1$$

eigenvectors of matrix A are vectors that change only length or sign, but not direction, when acted upon by A

N x N

$$\begin{aligned} A\mathbf{e}_1 &= \lambda_1 \mathbf{e}_1 \\ A\mathbf{e}_2 &= \lambda_2 \mathbf{e}_2 \\ &\vdots \\ A\mathbf{e}_N &= \lambda_N \mathbf{e}_N \end{aligned}$$

$$A = E \Lambda E^T$$

E: matrix whose columns are eigenvectors

Λ: diagonal matrix whose diagonal elements are eigenvalues

MATLAB Code:

```
[E, L] = eig(A)
```

`E` is matrix whose columns are eigenvectors
`L` is matrix whose diagonals are corresponding eigenvalues

Python Code:

```
E, L = np.linalg.eig(A)
```

`E` is matrix whose columns are eigenvectors
`L` is 1-D array whose elements are corresponding eigenvalues

fun facts

- eigenvalues and eigenvectors of matrix are *invariant* to change of basis.
  - eigenvalues will be identical
  - eigenvectors will be same vectors, just rewritten in new coordinates
- if A is symmetric ($A = A^T$), then eigenvalues are real and eigenvectors are orthogonal.
- eigenvalues and eigenvectors often have special meaning:
  - e.g., in PCA related to directions of maximum variance
  - e.g., in dynamical systems related to system stability

DYNAMICAL SYSTEMS

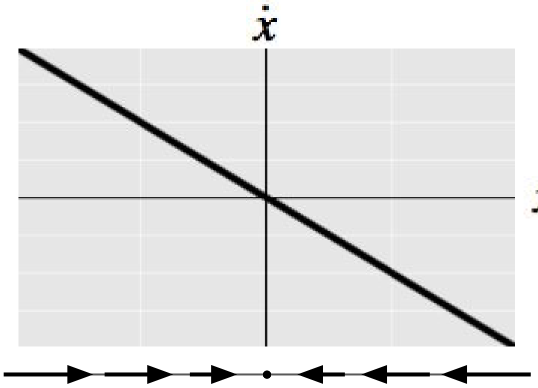
- "dynamical system" = system that changes in time
- represent with system of differential equations
- solving DEs is hard -- what can we learn about system without finding explicit solution?

1D	2D	ND
description	description	description
$\dot{x} = f(x)$ note: $\dot{x} \equiv \frac{dx}{dt}$	$\dot{x}_1 = f_1(x_1, x_2)$ $\dot{x}_2 = f_2(x_1, x_2)$ $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$	$\dot{x}_1 = f_1(x_1, \ldots, x_N)$ $\vdots$ $\dot{x}_N = f_N(x_1, \ldots, x_N)$ $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$
linear system	linear system	linear system
$\dot{x} = ax$	$\dot{x}_1 = a_{11}x_1 + a_{12}x_2$ $\dot{x}_2 = a_{21}x_1 + a_{22}x_2$ $\dot{\mathbf{x}} = A\mathbf{x}$	$\dot{x}_1 = a_{11}x_1 + \cdots + a_{1N}x_N$ $\vdots$ $\dot{x}_N = a_{N1}x_1 + \cdots + a_{NN}x_N$ $\dot{\mathbf{x}} = A\mathbf{x}$

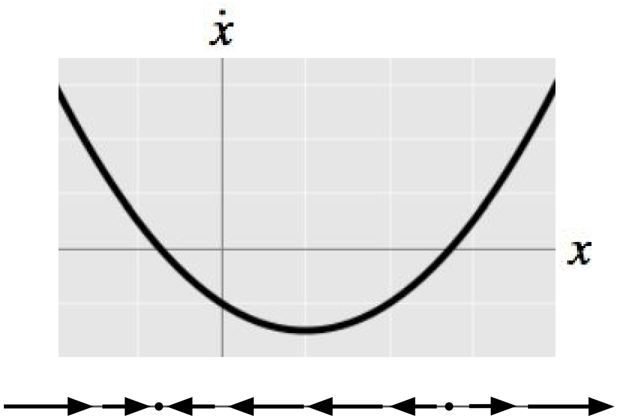
1D

graphical representation

linear

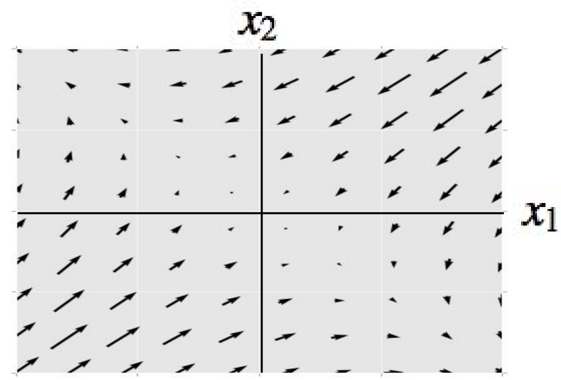


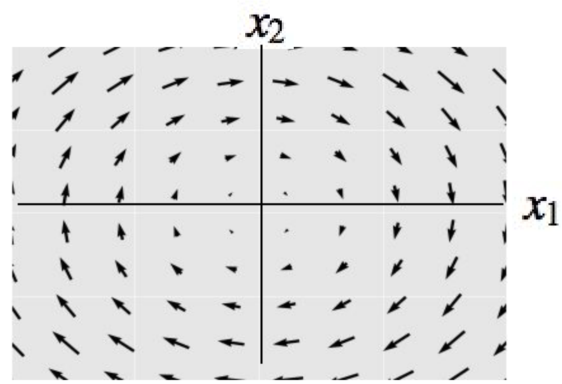
nonlinear



2D

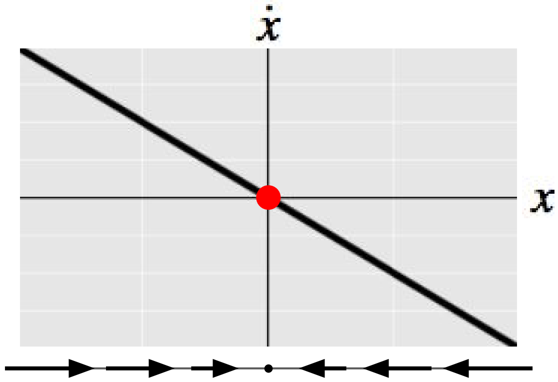
graphical representation (phase portrait)

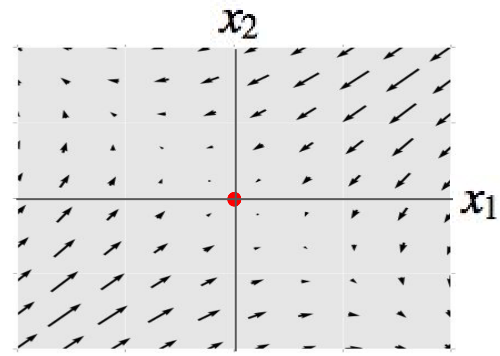




FIXED POINT ATTRACTORS

1D	2D	ND
fixed points	fixed points	



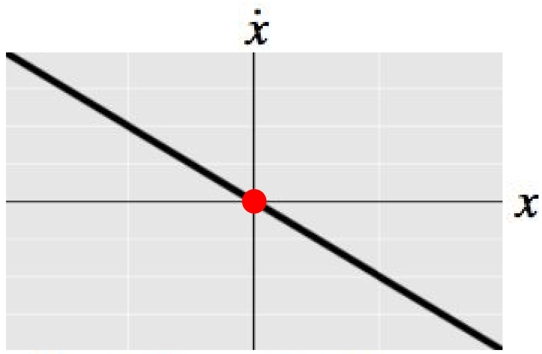


$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \mathbf{0}$$

1D

stability analysis

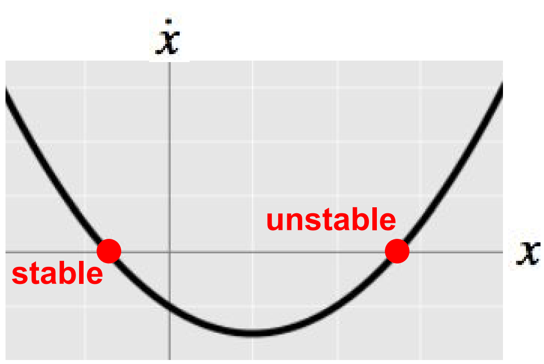
linear



slope > 0 : *unstable*

slope < 0 : *stable*

nonlinear



$\frac{df}{dx} > 0$: *unstable*

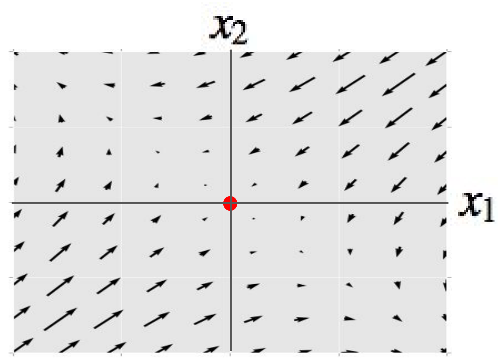
$\frac{df}{dx} < 0$: *stable*

(evaluated at fixed point)

ND

stability analysis

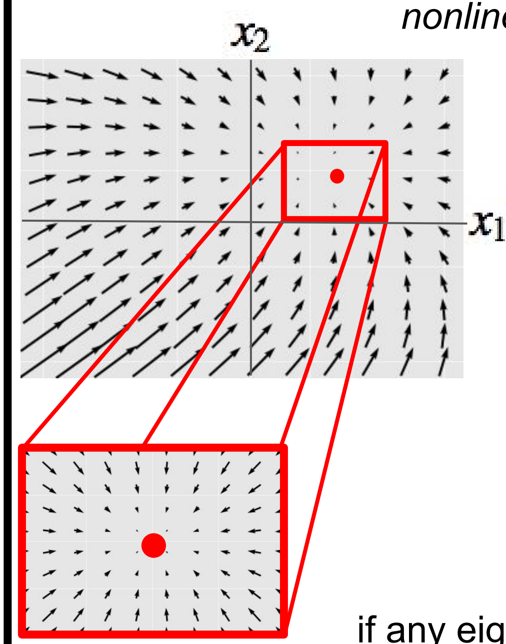
linear



$$\dot{\mathbf{x}} = A\mathbf{x}$$

if real part of any *eigenvalue* of A is greater than 0, fixed point is unstable

nonlinear



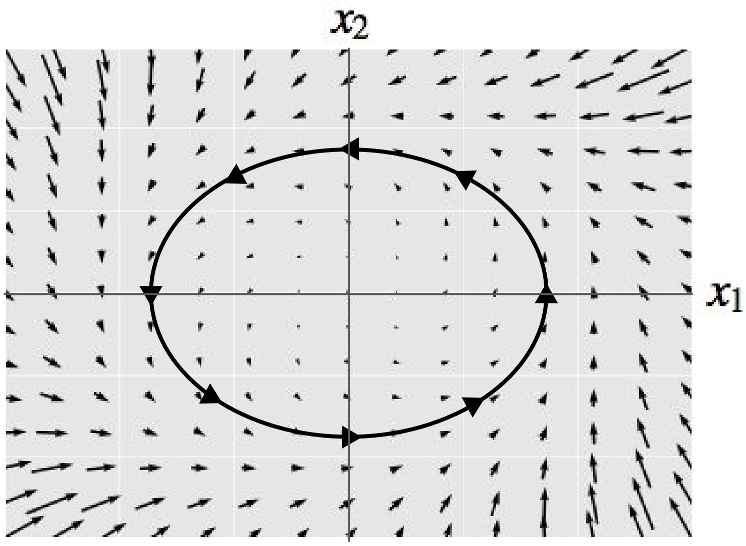
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1} & \cdots & \frac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

if any eigenvalue of **J** (evaluated at fixed point) is greater than 0, fixed point is unstable

OTHER ATTRACTORS

limit cycles



strange attractors (chaos)