Maximum Likelihood
and
Expectation
Maximization

Probability Basics

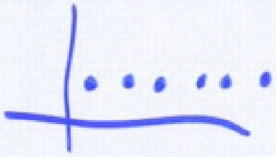
Ex

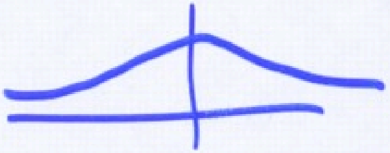
Ex

Sample Space $\{1, \dots, 6\}$ $\mathbb{R}$

Distribution $p_1 \cdots p_6 \geq 0, \sum p_i = 1$ p.d.f $f(x) \geq 0, \int_{R} f(x)\,dx = 1$

eg $p_1 = \cdots = p_6 = \frac{1}{6}$ $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$





Population vs Sample

population mean $\mu = \sum i\, p_i$ $\mu = \int x f(x)\,dx$

population Variance $\sigma^2 = \sum (i-\mu)^2 p_i$ $\sigma^2 = \int (x-\mu)^2 f(x)\,dx$

sample mean $\sum x_i / n$

Sample Variance $\sum (x_i - \bar{x})^2 / n$

Parameter Estimation

- Assuming sample $x_1, x_2, ..., x_n$ is from a parametric distributionf(x| theta), estimate theta.
- E.g.:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\theta = (\mu, \sigma^2)$$

Maximimum Likelihood Estimation

One (of many) approaches to parameter est.

Likelihood of $x_1 \cdots x_n$ =

$$L(x_1, \ldots, x_n | \theta) = \underbrace{\prod_{i=1}^{n}}_{\text{assume indp.}} f(x_i | \theta)$$

View this as a function of θ

what θ **maximizes** the likelihood

Typical approach: $\frac{\partial}{\partial \theta} L(x | \theta) = 0$

or $\frac{\partial}{\partial \theta} \ln L(x | \theta) = 0$

Example

$X_1 \cdots X_n$ coin flips ; θ = prob of heads

n_0 tails, n_1 heads, $n_0 + n_1 = n$

$$L(x_1 \cdots x_n | \theta) = (1-\theta)^{n_0} (\theta)^{n_1}$$

$$\ln L = n_0 \ln(1-\theta) + n_1 \ln \theta$$

$$\frac{d}{d\theta} \ln L = \frac{-n_0}{1-\theta} + \frac{n_1}{\theta} = 0$$

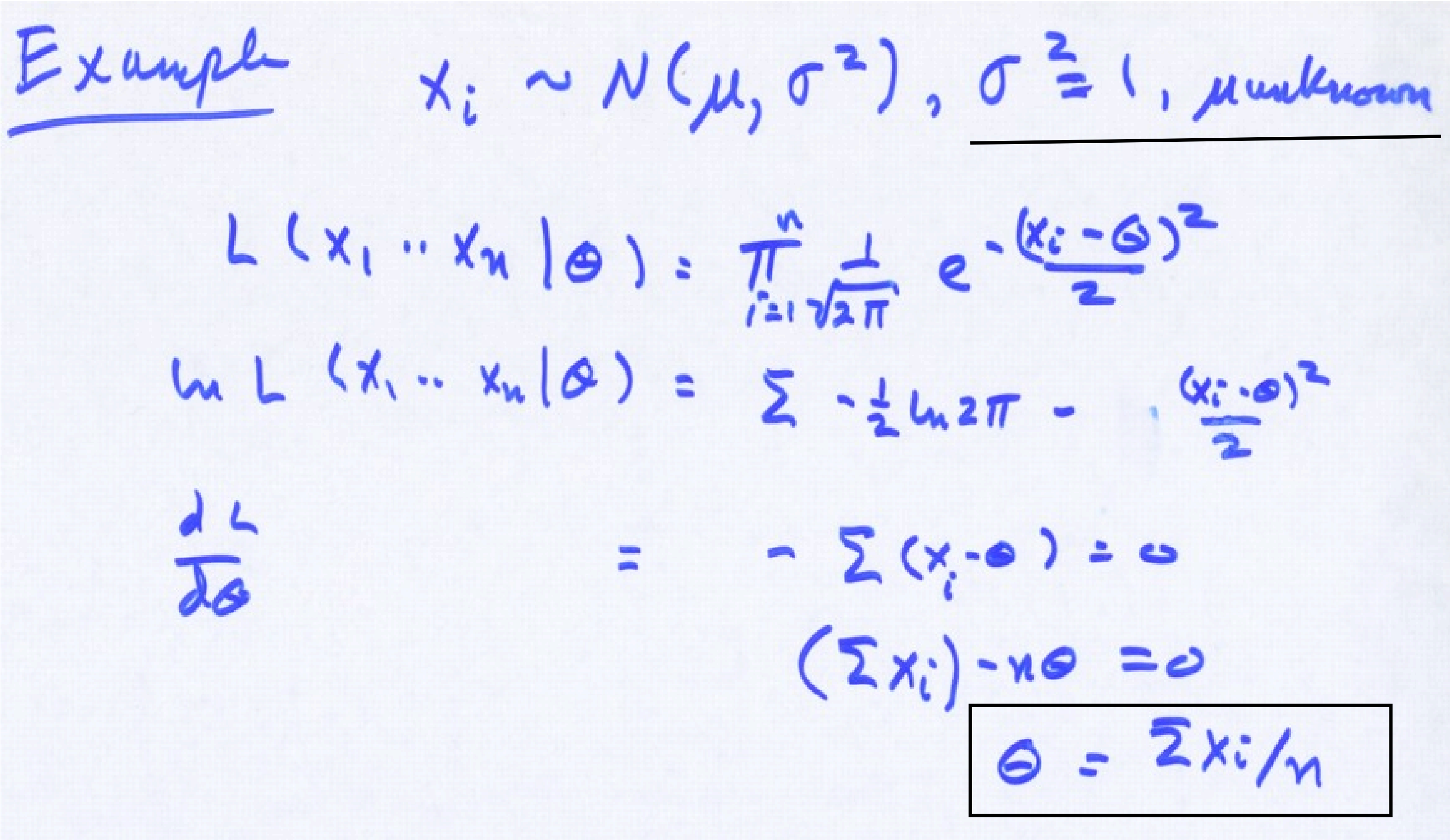
$$n_0 \theta = n_1 (1-\theta)$$

$$(n_0 + n_1)\theta = n_1$$

$$\boxed{\theta = \frac{n_1}{n}}$$

And verify it's max, not min

& not better on boundary



Example $x_i \sim N(\mu, \sigma^2)$, $\sigma^2 = 1$, μ unknown

$$L(x_1 \cdots x_n | \Theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_i - \Theta)^2}{2}}$$

$$\ln L(x_1 \cdots x_n | \Theta) = \sum -\frac{1}{2}\ln 2\pi - \frac{(x_i - \Theta)^2}{2}$$

$$\frac{dL}{d\Theta} = -\sum (x_i - \Theta) = 0$$

$$\left(\sum x_i\right) - n\Theta = 0$$

$$\boxed{\Theta = \sum x_i / n}$$

And verify it's max, not min
& not better on boundary

Example $x_i \sim N(\mu, \sigma^2)$, both unknown

...

$$\ln L(x_1 .. x_n | \theta_1 \theta_2) = \sum -\frac{1}{2} \ln 2\pi\theta_2 - \frac{(x_i - \theta_1)^2}{2\theta_2}$$

$$\frac{\partial}{\partial \theta_1} \ln L(x_1 .. x_n | \theta_1 \theta_2) = \sum \frac{x_i - \theta_1}{\theta_2} = 0$$

$$\Rightarrow \boxed{\theta_1 = \Sigma x_i / n}$$

$$\frac{\partial}{\partial \theta_2} \ln L(x_1 .. x_n | \theta_1 \theta_2) = \sum -\frac{2\pi}{2 \cdot 2\pi\theta_2} + \frac{(x_i - \theta_1)^2}{2\theta_2^2}$$

$$\sum -\frac{1}{2\theta_2} + \frac{(x_i-\theta_1)^2}{2\theta_2^{2}} = 0$$

$$\sum (x_i-\theta_1)^2 = n\theta_2$$

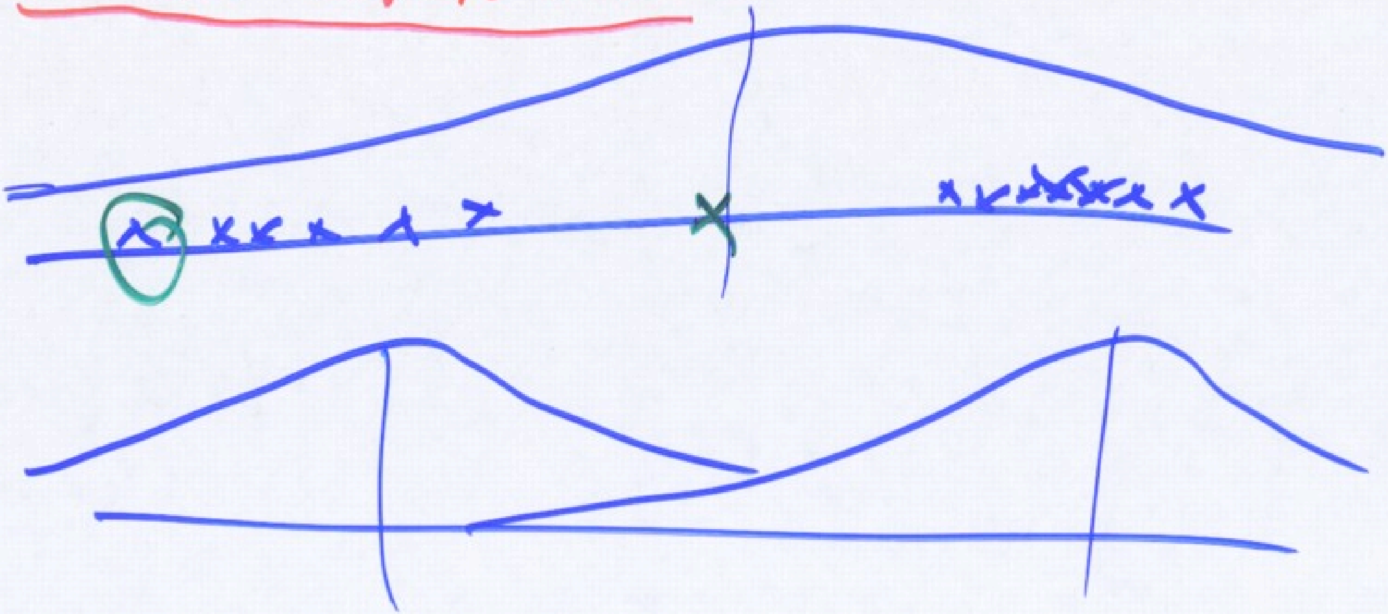
$$\boxed{\theta_2 = \sum (x_i-\theta_1)^2 / n}$$

A Biased (but consistent) estimate of population variance

An Example of <u>Overfitting</u>

Unbiased estimate: $\displaystyle\sum_{i=1}^{n} \frac{(x_i-\theta_1)^2}{n-1}$

A More Complex Problem



2 distributions $f_1(x)$, $f_2(x)$

$f_1(x|\theta)$, $f_2(x|\theta_2)$

Mixing parameter τ_1 τ_2

$\tau_1 + \tau_2 = 1$

Likelihood

$$L(x_1 \ldots x_n \mid \overset{\Theta}{\tau_1 \tau_2 \mu_1 \mu_2 \sigma_1^2 \sigma_2^2 \ldots})$$

$$= \prod_{i=1}^{n} \sum_{j=1}^{2} \tau_j \, f_j(x_i \mid \theta)$$

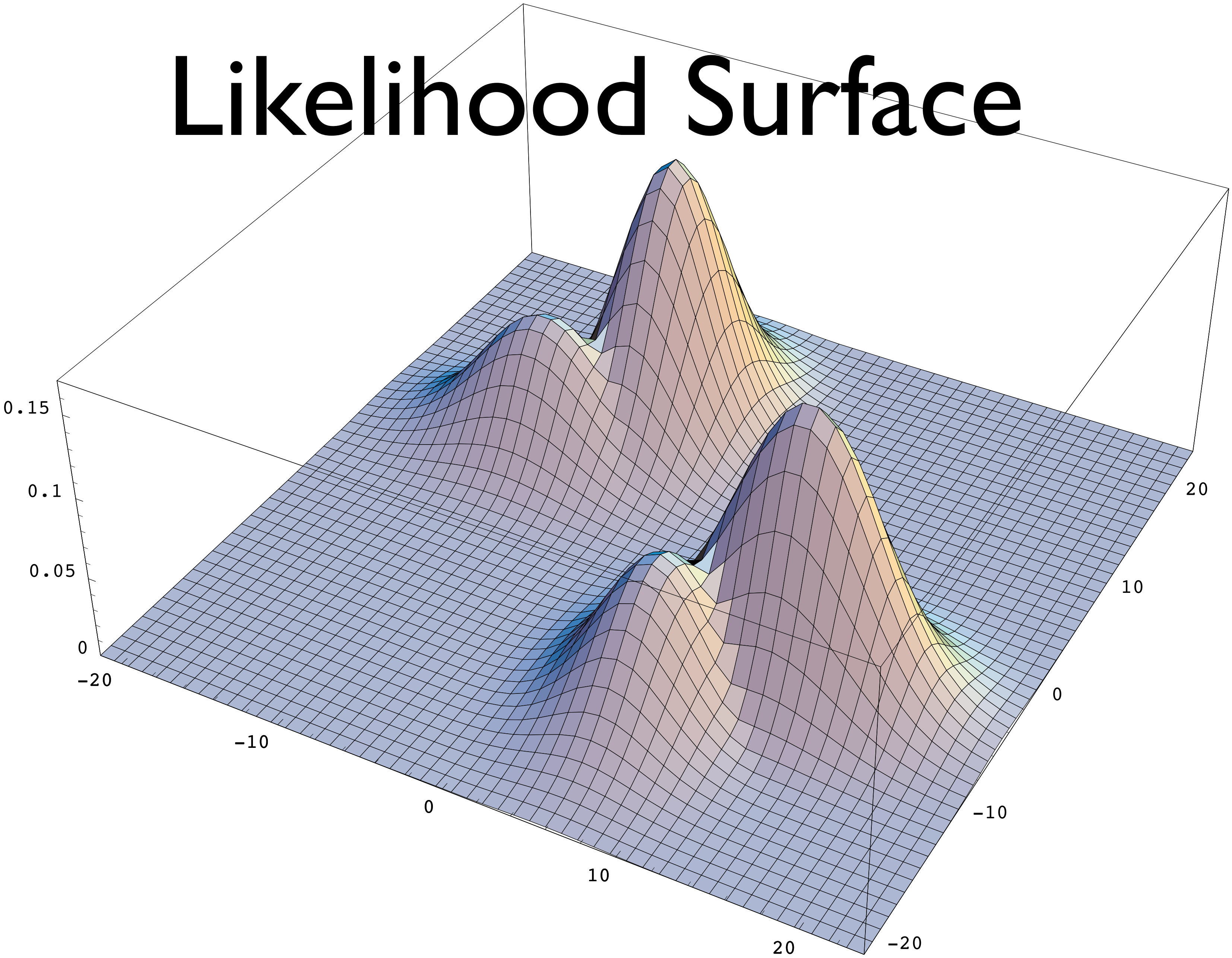
Probably too messy for closed-form solution

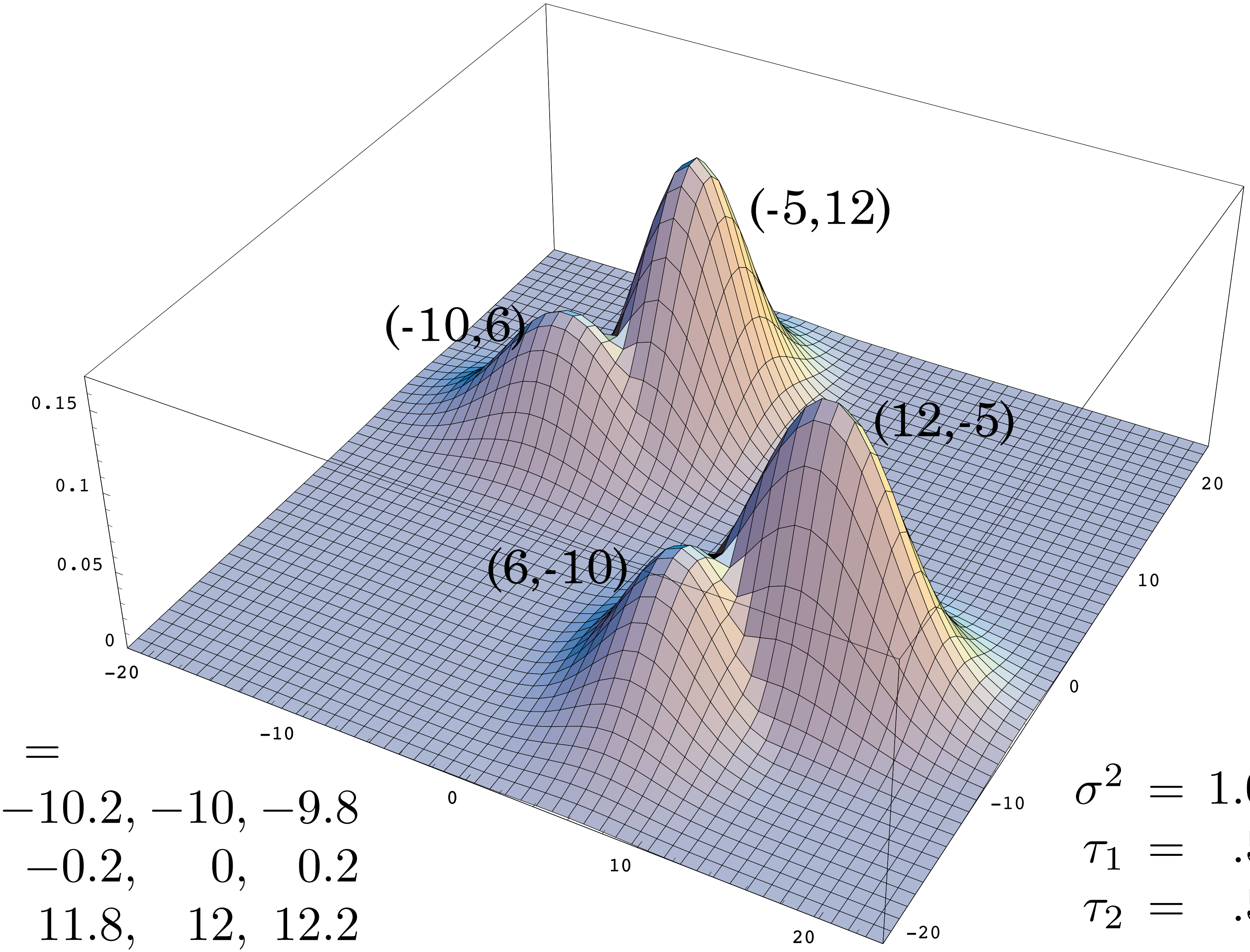
Full data

$$\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \quad \underbrace{\begin{array}{cc} z_{11} & z_{12} \\ z_{21} & z_{22} \\ z_{31} & z_{32} \end{array}}_{\text{Hidden Variables}}$$

$$z_{ij} \begin{cases} 0 \\ 1 & \text{iff } x_i \text{ comes from distribution } j \end{cases}$$

Likelihood Surface





$$x_i =$$

$$\begin{array}{rrr} -10.2, & -10, & -9.8 \\ -0.2, & 0, & 0.2 \\ 11.8, & 12, & 12.2 \end{array}$$

$$\begin{aligned} \sigma^2 &= 1.0 \\ \tau_1 &= .5 \\ \tau_2 &= .5 \end{aligned}$$

assume π_j θ_j fixed

A event that x_i drawn from f_1

B " " " " " " f_2

D "data" x_i is observed

$P(A|D)$

$P(D|A)$ Bayes rule

$$P(A|D) = \frac{P(D|A)\,P(A)}{P(D)}$$

$$P(D) = P(D|A)\cdot P(A) + P(D|B)\cdot P(B)$$

$$f_1(x_i|\theta_1) \quad \pi_1 \quad f_2(x_i|\theta_2) \quad \pi_2$$

Expected value of z_{i1}

M Step

$L(X_1 Z_{11} Z_{12} X_2 Z_{21} Z_{22} \cdots | \theta_T)$

X_i's known

If Z_{ij} known, then MLE θ_T easy

But we don't.

Instead maximize **expected** likelihood of visible data

$E(L(X_1, X_2, \ldots X_n | \theta_T))$

where Expectation is over distribution of hidden values (Z_{ij}'s)

$L(\vec{X}, \vec{Z} \mid \Theta \tau)$

$$\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\sum_{j=1}^{2} z_{ij}\cdot(x_i-\mu_j)^2}$$

$E(\ln L(X Z \mid \Theta \tau)) =$

$$E\left[\sum_{i=1}^{n} -\frac{1}{2}\ln 2\pi\sigma^2 - \frac{1}{2\sigma^2}\sum_{j=1}^{2} z_{ij}\,(x_i-\mu_j)^2 \right]$$

$$= \sum_{i=1}^{n} -\frac{1}{2}\ln 2\pi\sigma^2 \underbrace{- \frac{1}{2\sigma^2}\sum_{j=1}^{2} E(z_{ij})\,(x_i-\mu_j)^2}$$

Find μ_j maximizing ↑ using $E(z_{ij})$ From E-step.