CSE 525: Randomized Algorithms and Probability **Spring 2025**

HW4

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Instructions

- You should think about each problem by yourself for at least an hour before choosing to collaborate with others.
- You are allowed to collaborate with fellow students taking the class in solving the problems. But you **must** write your solution on your own.
- You are not allowed to search for answers or hints on the web. You are encouraged to contact the instructor or the TAs for a possible hint.
- You cannot collaborate on Extra credit problems
- Solutions typeset in LATEX are preferred.
- Feel free to use the Discussion Board or email the instructor or the TAs if you have any questions or would like any clarifications about the problems.
- Please upload your solutions to Gradescope.

In solving these assignments, feel free to use these approximations:

$$1 - x \approx e^{-x}, \qquad \sqrt{1-x} \approx 1 - x/2, \qquad n! \approx (n/e)^n, \qquad \left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{en}{k}\right)^k$$

The problems of this HW are hard. To get full points it is enough to solve one of the following problems, although you are encouraged to try both.

1) Let μ be the uniform distribution over all spanning trees of a given graph $G = (V, E)$. For a set $S \subseteq V$, let $q := |S| - 1 - \mathbb{E}_\mu[|E(S) \cap T|]$, where $E(S) = \{(u,v) \in E : u, v \in S\}$ denotes the set of edges between vertices of S. Then, for any $B \subseteq E \setminus E(S)$, we have

$$\mathbb{E}_\mu[|B \cap T|] - q \le \mathbb{E}_\mu[|T \cap B| \mid T \text{ induces a tree in } S] \le \mathbb{E}_\mu[|B \cap T|]$$

Hint: Recall that every spanning tree of G has exactly $|V| - 1$ edges.

2) In the determinant maximization problem we are given n vectors $v_1, \ldots, v_n \in \mathbb{R}^d$ and an integer k and we want to output a set $S \in \binom{n}{k}$ maximizing $\det(\sum_{i \in S} v_i v_i^T)$. We use the following concave program:

$$\begin{aligned} \max \quad & \log \sum_{S \in \binom{n}{k}} x^S \det(\sum_{i \in S} v_i v_i^T) \\ \text{s.t.,} \quad & \sum_i x_i = k, \qquad x_i \ge 0, \forall i. \end{aligned} \tag{1.1}$$

This program is concave since any real stable polynomial is log-concave. Use this program to design a randomized algorithm that outputs a set $S \in \binom{n}{k}$ such that

$$\mathbb{E}\left[\det\left(\sum_{i \in S} v_i v_i^T\right)\right] \geq e^{-k} OPT.$$