CSE 525: Randomized Algorithms **Spring 2025**

Lecture 9: Applications of SR Distributions

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

9.1 Gurvits' Machinery

Given a matrix $M \in \mathbb{R}_{\geq 0}^{n\times n}$ the permanent of M is defined as follows:

$$\text{per}(M) = \sum_{\sigma\in\mathbb{S}_n}\prod_{i=1}^{n} M_{i,\sigma_i}.$$

This quantity is #P-hard to compute exactly so one can only appeal to approximation algorithms. The problem has been studied for years. After much effort, Jerrum, Sinclair and Vigoda [**JSV04**] managed to design a randomized $1\pm\epsilon$ approximation algorithm for this problem that runs in time polynomial in $n, 1/\epsilon$.

Designing a deterministic algorithm with the same approximation factor remains a *fundamental open problem*. This question relates to several fundamental problem in TCS under the umbrella of polynomial identity testing (PIT). It is believed that one can simulate any polynomial time randomized algorithm with a polynomial time deterministic algorithm. Permanent is perhaps the most fundamental problem which we have no deterministic approach for.

After a long line of works–see Linial, Samarodnitsky, Widgerson [**LSW**], Barvinok [**Bar97**], Gurvits [**GS02**; **Gur06**], Gurvits, Samarodnitsky [**GS14**] and Anari, Rezaei [**AR18**]–the best deterministic approximation factor known is $(\sqrt{2})^n$. Note that if we can design a 2^{n^ϵ} approximation algorithm for any $\epsilon > 0$ then it is possible to improve the approximation factor to $1\pm\epsilon$.

In this section, we will prove the following result of Gurvits:

Theorem 9.1 (Gurvits [**Gur06**])**.** *Let $p \in \mathbb{R}[z_1,\dots,z_n]$ be a real stable polynomial with non-negative coefficients. Then,*

$$e^{-n}\inf_{z>0}\frac{p(z_1,\dots,z_n)}{z_1\dots z_n} \leq \partial_{z_1}\dots\partial_{z_n}p\big|_{z=0} \leq \inf_{z>0}\frac{p(z_1,\dots,z_n)}{z_1\dots z_n}.$$

In other words, the above theorem shows that we can approximate the coefficient of the monomial $z_1\dots z_n$ in the real stable polynomial p (with non-negative coefficient) with the mathematical program $\inf_{z>0}\frac{p(z_1,\dots,z_n)}{z_1\dots z_n}$.

Convexity. First we show that $\inf_{z>0}\frac{p(z_1,\dots,z_n)}{z_1\dots z_n}$ can be formulated as a convex programming problem. The idea is to do a change of variables $e^{y_i}\leftrightarrow z_i$. Note that since $z_i > 0$ this is a valid change of variables. Furthermore to minimize the ratio we can equivalently minimize the log of the ratio. So, we claim,

$$\log\frac{p(e_1^y,\dots,e^{y_n})}{e^{y_1}\dots e^{y_n}} = \log p(e_1^y,\dots,e^{y_n}) - \sum_{i=1}^{n} y_i$$

is convex. To see this, it is enough to show $\log p(e^{y_1}, \dots, e^{y_n})$ is convex, as the second term is linear. To see this, recall that log-sum-exp is a convex function i.e.,

$$\log \sum_{i=1}^{n} a_i e^{\langle b_i, y \rangle}$$

is convex as long as $a_1, \dots, a_n \geq 0$ and $b_i \in \mathbb{R}^n$ for all i.

9.1.1 Applications

Now, we show how to use this theorem to get an e^n approximation for permanent. Consider the following polynomial

$$p(z_1, \dots, z_n) = \prod_{i=1}^{n} \sum_{j=1}^{n} M_{i,j} z_j. \tag{9.1}$$

First, observe that p is real stable since all entries of M are nonnegative and the product of real stable polynomials is real stable. Second, observe the coefficient of the monomial $z_1 \dots z_n$ is exactly $\text{per}(M)$, i.e.,

$$\partial_{z_1} \dots \partial_{z_n} p \mid_{z=0} = \text{per}(M).$$

Therefore, Theorem 9.1 gives an e^n approximation algorithm to the $\text{per}(M)$.

Next, we use this theorem to prove the van-der-Waerden conjecture:

Theorem 9.2. *Let $M \in \mathbb{R}^{n \times n}_{\geq 0}$ be a doubly stochastic matrix, i.e., the sum of the entries in every row and column is exactly 1. Then, $\text{per}(M) \geq e^{-n}$.*

Proof. By Theorem 9.1 we can write

$$\text{per}(M) \geq e^{-n} \inf_{z>0} \frac{p(z_1, \dots, z_n)}{z_1 \dots z_n}$$

where p is the same polynomial as in (9.1). So it is enough to show $\inf_{z>0} \frac{p(z_1, \dots, z_n)}{z_1 \dots z_n} \geq 1$. To see this, we use the weighted AM-GM inequality.

Theorem 9.3 (Weighted AM-GM Inequality)**.** *Let $a_1, \dots, a_n \geq 0$ and $\lambda_1, \dots, \lambda_n \geq 0$ such that $\sum_i \lambda_i = 1$. Then,*

$$\sum_i \lambda_i a_i \geq \prod_i a_i^{\lambda_i}.$$

By the above theorem we can write

$$p(z_1, \dots, z_n) = \prod_{i=1}^{n} \sum_{j=1}^{n} M_{i,j} z_j \geq \prod_{i=1}^{n} \prod_{j=1}^{n} z_j^{M_{i,j}} = \prod_{j=1}^{n} z_j^{\sum_{i=1}^{n} M_{i,j}} = \prod_{j=1}^{n} z_j,$$

as desired. The inequality follows by (weighted) AM-GM and the fact that every row of M adds up to 1 and the last equality follows by the fact that every column of M adds up to 1. $\square$

9.1.2 Proof of Theorem

Before proving Theorem 9.1 let us first gain some intuition. The right side of the inequality is trivial and it holds for any polynomial with non-negative coefficients. This is because if we let $p = \sum_{\alpha \in \mathbb{N}_{\geq 0}} c_\alpha z^\alpha$, then

$$\inf_{z>0} \frac{p(z_1, \ldots, z_n)}{z_1 \ldots z_n} = c_{1,1,\ldots,1} + \inf_{z>0} \sum_{\alpha \in \mathbb{N}_{\geq 0}} c_\alpha \frac{z^\alpha}{z_1 \ldots z_n} \geq c_{1,1,\ldots,1},$$

as desired.

Now, the left side of the inequality crucially uses the fact that p is real stable. For example, consider the non-stable polynomial $z_1^2 + z_2^2$. Then,

$$\inf_{z>0} \frac{z_1^2 + z_2^2}{z_1 z_2} = \inf_{z>0} \frac{z_1}{z_2} + \frac{z_2}{z_1} \geq 2,$$

where as the coefficient of $z_1 z_2$ is 0.

We start the proof of Theorem 9.1 by proving a univariate version of the theorem. This not only will serve as the base of the induction, but will also give us some clues on how to approach the main theorem.

Lemma 9.4. *For any real rooted polynomial $f \in \mathbb{R}[t]$ with nonnegative coefficients,*

$$f'(0) \geq \frac{1}{e} \inf_{t>0} f(t)/t.$$

Proof. First, we show that f is a log-concave function.

Fact 9.5. *Let $f \in \mathbb{R}[t]$ be a real rooted polynomial with nonnegative coefficients then $\log f$ is a concave function of $\mathbb{R}_{\geq 0}$.*

This is because (dropping the constant term for convenience):

$$\log f = \log \prod_{i=1}^{n} (t + \alpha_i) = \sum_{i=1}^{n} \log(t + \alpha_i),$$

where $-\alpha_i$'s are roots of f. Since f has all positive coefficients, it is non-zero over the positive reals so $\alpha_i \geq 0$ for all i. The claim follows because log is a concave function and the sum of concave functions is concave.

Next, we prove the lemma. Since $\log f$ is concave, for any $t \geq 0$ we can write,

$$\log f(t) \leq \log f(0) + t(\log f(0))'$$

Therefore, for any $t \geq 0$ we get

$$\log \frac{f(t)}{t} \leq \log f(0) + t \frac{f'(0)}{f(0)} - \log t$$

Setting $t = f(0)/f'(0)$ in the RHS we get

$$\inf_{t>0} \log \frac{f(t)}{t} \leq \log f(0) + 1 - \log \frac{f(0)}{f'(0)} = 1 + \log f'(0)$$

as desired. $\square$

Having proved the lemma we are ready to prove Theorem 9.1. We prove by induction on n. Let $q(z_1, \ldots, z_{n-1}) = \partial_{z_n} p|_{z_n=0}$. Note that by closure properties of real stable polynomials q is a real stable polynomial. Furthermore, it has non-negative coefficients. Then,

$$\partial_{z_1} \ldots \partial_{z_n} p|_{z=0} = \partial_{z_1} \ldots \partial_{z_{n-1}} q|_{z=0} \geq e^{-(n-1)} \inf_{z_1,\ldots,z_{n-1}>0} \frac{q(z_1, \ldots, z_{n-1})}{z_1 \ldots z_{n-1}} \tag{9.2}$$

where the inequality follows by IH. Say the infimum in the RHS is attained at a point $z_1^*, \ldots, z_{n-1}^*$[1]. Define $f(z_n) = p(z_1^*, \ldots, z_{n-1}^*, z_n)$. Then, observe that

$$q(z_1^*, \ldots, z_{n-1}^*) = f'(0) \geq \frac{1}{e} \inf_{z_n>0} \frac{f(z_n)}{z_n} = \frac{1}{e} \inf_{z_n>0} \frac{p(z_1^*, \ldots, z_{n-1}^*, z_n)}{z_n}, \tag{9.3}$$

where the inequality follows by the fact that f is a real stable polynomial with non-negative coefficients. Therefore,

$$\partial_{z_1} \ldots \partial_{z_n} p|_{z=0} \geq e^{-(n-1)} \frac{q(z_1^*, \ldots, z_{n-1}^*)}{z_1^* \ldots z_{n-1}^*} \geq e^{-n} \inf_{z_n>0} \frac{p(z_1^*, \ldots, z_{n-1}^*, z_n)}{z_1^* \ldots z_{n-1}^* z_n} \geq e^{-n} \inf_{z>0} \frac{p(z_1, \ldots, z_n)}{z_1 \ldots z_n}$$

where the first inequality follows by (9.2) and the second inequality follows by (9.3). This finishes the proof of Theorem 9.1.

Remark 9.6. *The above proof is fairly general; it works for any polynomial whose univariate restrictions are log-concave. Gurvits has studied many generalizations and extensions of the above proof such as using it to estimate mixed volume of convex bodies [**Gur06**] .*

9.2 Applications to TSP

Let μ be a uniform distribution over spanning trees of a graph $G = (V, E)$. Furthermore suppose that $\mathbb{E}\left[[] \, d_T(v)\right] = 2$ for all vertices $v \in G$. Such a distribution can be constructed efficiently using the linear programming relaxation of TSP.

Very recently, in a joint work with Anna Karlin and Nathan Klein we proved the following theorem:

Theorem 9.7 ([**KKO20**])**.** *For some $\epsilon \geq 10^{-36}$, the maximum entropy algorithm gives a $1.5 - \epsilon$ approximation algorithm for TSP.*

The proof uses many properties of Strongly Rayleigh distribution μ.

Next, we will explain a crucial property of such a distribution and the generalization which is used in our work.

Lemma 9.8. *Let $v_1, \ldots, v_k$ be vertices of G such that the induced graph $G[\{v_1, \ldots, v_k\}]$ has no edges. Then,*

$$\mathbb{P}_{T\sim\mu}\left[d_T(v_1) = \cdots = d_T(v_k) = 2\right] \geq e^{-k}$$

where $d_T(v)$ is the degree of a vertex v in the sampled tree T.

Proof. Let $S_1, \ldots, S_k$ be the set of edges incident to $v_1, \ldots, v_k$ respectively and let F be the rest of the edges. Note that since $v_1, \ldots, v_k$ do not share edges, $S_1, \ldots, S_k$ are mutually disjoint. Define

$$p(y_1, \ldots, y_k) = g_{\mu*\lambda}\left(\begin{cases} z_e = y_1 & \forall e \in S_1, \\ \ldots & \\ z_e = y_k & \forall e \in S_k \\ z_e = 1 & \text{otherwise} \end{cases}\right).$$

[1]to be more precise we need to do a ϵ-δ argument but here we avoid that for the simplicity of the argument.

Note that in this definition we crucially use that $S_1, \ldots, S_k$ are disjoint. By closure properties of real stable polynomials p is real stable. We can re-write p as follows:

$$p(y_1, \ldots, y_k) = \sum_T \mu(T) \prod_{i=1}^k y_i^{d_T(v_i)}.$$

It follows that

$$\mathbb{P}\left[d_T(v_1) = \cdots = d_T(v_k) = 2\right] = 2^{-k} \partial_{y_1}^2 \ldots \partial_{y_k}^2 p|_{y=0},$$

i.e., the RHS is the coefficient of $y_1^2 \ldots y_k^2$ in p. Furthermore, note that each of the vertices $v_1, \ldots, v_k$ have degree at least 1 in T; so we can factor out a monomial $y_1 \ldots y_k$,

$$p(y) = y_1 \ldots y_k q(y_1, \ldots, y_k).$$

It follows that q is also real stable. So, we need to show that

$$\partial_{y_1} \ldots \partial_{y_k} q|_{y=0} \geq e^{-k}.$$

Since q is real stable and has non-negative coefficients, by Gurvits' Theorem 9.1, we have

$$\partial_{y_1} \ldots \partial_{y_k} q|_{y=0} \geq e^{-k} \inf_{y>0} \frac{q(y)}{y_1 \ldots y_k}.$$

So, all we need to show is that

$$\inf_{y>0} \frac{q(y)}{y_1 \ldots y_k} \geq 1. \tag{9.4}$$

First, observe that we can write q as follows:

$$\begin{aligned} q(y_1, \ldots, y_k) &= \sum_T \mu(T) \prod_{i=1}^k y_i^{d_T(v_i)-1} \\ &\geq \prod_T \left(\prod_{i=1}^k y_i^{d_T(v_i)-1} \right)^{\mu(T)} \\ &= \prod_{i=1}^k y_i^{\sum_T \mu(T) d_T(v_i) - 1} = \prod_{i=1}^k y_i^{2-1}, \end{aligned}$$

where the inequality follows by weighted AM-GM and the last identity follows by the fact that $\mathbb{E}\left[d_T(v)\right] = 2$ for any vertex. This proves (9.4). $\square$

We proved the following generalization of the above fact in [**KKO20**].

Theorem 9.9. *Given a SR distribution $\mu : 2^{[n]} \to \mathbb{R}_+$, and disjoint sets $A_1, \ldots, A_k$ such that for any $S \subseteq [k]$,*

$$\mathbb{P}_{T \sim \mu}\left[\sum_{i \in S} |T \cap A_i| = |S|\right] \geq \epsilon$$

Then,

$$\mathbb{P}\left[\forall i : |T \cap A_i| = 1\right] \geq \frac{\epsilon^{2^k+1}}{k!}.$$

In the exercises we will see the proof of the above theorem for $k = 2$. See a recent work of Gurvits and Leake [**GL20**] for a somewhat stronger bound.

9.3 Nash Welfare Maximization Problem

In the Nash welfare maximization problem we have a set of n agents $[n] := \{1, \ldots, n\}$ and a set of m items $[m] := \{1, \ldots, m\}$ (where assume that $m \gg n$; we further assume that agent i has value $v_{i,j} \geq 0$ for item j. For an assignment of items to agents $x : [n] \times [m] \to \{0, 1\}$ the value of i is defined as

$$v_i(x) := \sum_j x_{i,j} v_{i,j}.$$

The goal is to find an assignment to maximize the geometric mean of the values of all agents, i.e.,

$$\max_x \left(\prod_{i=1}^n v_i(x) \right)^{1/n}.$$

In this section we use the following generalization of Gurvits' machinery due to Anari-Oveis Gharan '17 to give a constant factor approximation algorithm for this problem.

Given two polynomials $p, q \in \mathbb{R}_{\geq 0}[z_1, \ldots, z_n]$ we define the inner product of p, q as follows:

$$\langle p, q \rangle = q(\partial z) p|_{z=0} = p(\partial z) q|_{z=0} = \sum_{\kappa \in \mathbb{Z}^n_{\geq 0}} c_p(\kappa) c_q(\kappa) \kappa!,$$

where $\kappa! = \prod_{i=1}^n \kappa_i!$.

Theorem 9.10. *For any two real stable polynomials $p, q \in \mathbb{R}_{\geq 0}[z_1, \ldots, z_n]$,*

$$\sup_{\alpha \in \mathbb{R}^n_{\geq 0}} e^{-\alpha} f(p, q, \alpha) \leq \langle p, q \rangle \leq \sup_{\alpha \in \mathbb{R}^n_{\geq 0}} f(p, q, \alpha) \tag{9.5}$$

where

$$f(p, q, \alpha) := \inf_{y, z > 0} \frac{p(y) q(z)}{(yz/\alpha)^\alpha}.$$

Furthermore, in the special case that p, q are multilinear we can improve $e^{-\alpha}$ to $(1-\alpha)^{1-\alpha}$.

We note that similar to Gurvits' capacity function $f(p, q, \alpha)$ can computed efficiently in polytime.

First, let me discuss a natural convex relaxation for this problem:

$$\begin{aligned}
\max \quad & \sum_{i=1}^n \log \sum_j v_{i,j} x_{i,j}, & \\
\text{s.t.,} \quad & \sum_{i=1}^n x_{i,j} = 1 & \forall 1 \leq j \leq m \\
& x_{i,j} \geq 0 & \forall i, j.
\end{aligned} \tag{9.6}$$

Note that the optimum solution is obviously a feasible solution to the above program, so above is a convex relaxation of the problem. It turns out that the above program has an infinite integrality gap. For example, suppose we have n items where the first one is a BMW which has value 1 for everyone and the rest of them are chocolates and cookies which have value ϵ for everyone where $\epsilon > 0$ is very small. Obviously everyone very much like to get the BMW; but the integral only gets to assign BMW to one person and it gets a value of $\epsilon^{n-1/n}$. The CP fraction solutions assigns $1/n$ fraction of BMW to each agent and receives a value of at least $1/n$. As $\epsilon \to 0$ the value of integral optimum converges to 0.

Before discussing our convex program let us first discuss a natural rounding algorithm. Say $x : [n] \times [m] \to [0, 1]$ is a fractional assignment. A natural rounding algorithm assigns item j to agent i with probability $x_{i,j}$; note that since $\sum_i x_{i,j} = 1$ we have a valid probability distribution for every item j. Let $V_i(x)$ be the random variable of the value that i receives under this assignment. Then,

$$\mathbb{E}\left[V_i(x)\right] = \sum_j x_{i,j} v_{i,j}.$$

But, the value of our algorithm is the sum of values of agents, it is rather the product.

Lemma 9.11. *Let*

$$p_x(z_1, \dots, z_m) \quad = \quad \prod_{i=1}^{n} \sum_j v_{i,j} x_{i,j} z_j, \tag{9.7}$$

Then,

$$\mathbb{E}\left[\prod_i V_i(x)\right] = \sum_{S \in \binom{m}{n}} \prod_{j \in S} \partial_{z_j}(p)|_{z=0},$$

Proof. The lemma simply follows by the fact that any monomial of p that has a square of say z_j (for some j) corresponds to assigning the j-th item to two or more agents. □

Let

$$q(z_1, \dots, z_m) = e_n(z_1, \dots, z_m) = \sum_{S \in \binom{m}{n}} z^S.$$

Observe that

$$\sum_{S \in \binom{m}{n}} \prod_{j \in S} \partial_{z_j}(p_x)|_{z=0} = \langle p_x, q \rangle,$$

Now, observe that not only we can use Theorem 9.10 to estimate $\langle p_x, q \rangle$, since $f(p, q, \alpha)$ is a convex function we can change also optimize over x such that $\langle p_x, q \rangle$ is maximized. We use the following program:

$$\begin{aligned} \max \quad & f(p_x, q, \alpha) \\ \text{s.t.,} \quad & \sum_i x_{i,j} = 1 \quad \forall 1 \le j \le m, \\ & x_{i,j} \ge 0 \qquad\qquad \forall i, j. \end{aligned} \tag{9.8}$$

Given an optimal solution x to this program we simply run the above randomized rounding algorithm; by Theorem 9.10 the objective function of the above program gives an $e^{\min\{\deg p, \deg q\}}$ approximation to OPT^n. Since $\deg p = \deg q = n$ we indeed obtain an e approximation to OPT.

Finally, let me remark that $f(p_x, q, \alpha)$ is indeed a concave function (of α, x). This is because p_x is a log-concave function of x.