CSE 5215: Randomized Algorithms **Spring 2023**

Problem Set

Submit to gradescope

Instructions

- You should think about each problem by yourself for at least an hour before choosing to collaborate with others.
- You are allowed to collaborate with fellow students taking the class in solving the problems. But you **must** write your solution on your own.
- You are not allowed to search for answers or hints on the web. You are encouraged to contact the instructor or the TAs for a possible hint.
- You cannot collaborate on Extra credit problems
- Solutions typeset in LATEX are preferred.
- Feel free to use the Discussion Board or email the instructor or the TAs if you have any questions or would like any clarifications about the problems.
- Please upload your solutions to Gradescope.

All problems have 20 points.

In solving these assignments, feel free to use these approximations:

$$1 - x \approx e^{-x}, \qquad \sqrt{1-x} \approx 1 - x/2, \qquad n! \approx (n/e)^n, \qquad \left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{en}{k}\right)^k$$

1) **Submit by April 9.** Let $G = (V, E)$ be a bipartite graph with n vertices and a list $S(v)$ of at least $\log_2(n+1)$ colors associated with each vertex $v \in V$. Design a randomized polynomial time algorithm that finds a proper coloring of (vertices) of G assigning to each vertex v a color from its list $S(v)$ with probability at least $1 - 1/n$.

2) **Submit by April 9.** Show there is a positive number $c > 0$ (independent of n) such that the following holds. For any n, and any n real numbers $a_1, a_2, \ldots, a_n$ satisfying $a_1^2 + \cdots + a_n^2 = 1$, if $s_1, s_2, \ldots, s_n \in \{-1, +1\}$ are i.i.d. uniformly random signs, then

$$\mathbb{P}\left[\left|\sum_i s_i a_i\right| \le 1\right] \ge c.$$

Hint: Consider the case $\max\{a_1, , a_n\} < 1 - \Omega(1)$. separately.

3) **Send by April 16th** Given a d-regular undirected graph $G = (V, E)$ with labels $\ell : E \to [d]$ where $[d] = \{0, \ldots, d-1\}$. We say G is a consistently labelled if for any vertex $v \in V$, edges adjacent to v are labeled with $[d]$. Given a sequence $a_0, \ldots, a_{m-1} \in [d]$ and a starting vertex $v_0 \in V$ of a consistently labelled graph $G = (V, E)$, we can traverse G by first going to the unique neighbor of v_0, say v_1, connected with

the edge labelled a_0, then going to the unique neighbor of v_1, say v_2, connected with the edge labelled a_1, etc. We say a sequence $a_0, \dots, a_{m-1}$ **covers** G if for **any** starting vertex $v_0 \in V$, following this sequence we visit **all** vertices of G.

Given $1 \le d < n$, prove that there exists a sequence $a_0, \dots, a_{m-1}$ for some $m = \text{poly}(n, d)$ such that for **any** d-regular consistently labelled n vertex graph $G = (V, E)$ the sequence $a_0, \dots, a_{m-1}$ covers G.

You can use the following fact:

Theorem 1.1. *For any d regular graph and any vertex v, a simple random walk started at v visits all vertices of G in (at most) $2|E||V|$ steps in* **expectation**. *Recall that a simple random walk on G in each step moves to a uniformly random neighbor of the current vertex.*

4) **Submit by April 16th** Use the second moment method to show that there is a phase transition for connectivity in a $G(n,p)$ graph. Let $\mathcal{E}$ be the event that $G(n,p)$ is connected. Namely prove the following two facts:

   i) If $p \ll \frac{\log n}{n}$, then $\mathbb{P}[\mathcal{E}] \to 0$ as $n \to \infty$.
   Hint: Let $\mathcal{E}'$ be the event that $G(n,p)$ has an isolated vertex. Prove that $\mathbb{P}[\mathcal{E}'] \to 1$ as $n \to \infty$.

   ii) If $p \gg \frac{\log n}{n}$, then $\mathbb{P}[\mathcal{E}] \to 1$ as $n \to \infty$.

5) **Submit by April 23rd.** Let $G = (V, E)$ be a simple graph and suppose each $v \in V$ is associated with a set $S(v)$ of colors of size **at least** $|S(v)| \ge 10d$, where $d > 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are **at most** d neighbors u of v such that c lies in $S(u)$ (note that in principal there is no bound on the number of neighbors of v). Prove that there is a **proper coloring** (i.e., any two neighboring vertices must have distinct colors) of G assigning to each vertex v a color from its class $S(v)$.

6) **Submit by April 23rd.** Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be a set of bad events defined on independent variables $Z_1, \dots, Z_m$ with corresponding dependency graph G (ignore conditions of the LLL for this problem). Note that $\mathbb{P}[\neg \mathcal{A}_i]$ is uniquely defined as a function of the probabilities of Z_i's. Prove or disprove: In the Moser-Tardos algorithm, for a fixed $t > 1$ and $i \in [n]$, the probability that the t-th resampled event is $\mathcal{A}_i$ is at most $\mathbb{P}[\mathcal{A}_i]$. Prove this or find a counterexample.

7) **Submit by April 30th.** Given a bipartite graph $G = (X, Y, E)$ with $|X| = |Y| = n$, a perfect matching is a set $M \subseteq E$ of $|M| = n$ edges such that every vertex of X, Y is incident to exactly one edge of M. Prove or Disprove: For every bipartite graph $G = (X, Y, E)$ with $|X| = |Y| = n$ the uniform distribution over all perfect matchings of G is pairwise negatively correlated.

8) **Submit by April 30th.**Let μ be the uniform distribution over all spanning trees of a given graph $G = (V, E)$. For a set $S \subseteq V$, let $q := |S| - 1 - \mathbb{E}_\mu[|E(S) \cap T|]$, where $E(S) = \{(u,v) \in E : u, v \in S\}$ denotes the set of edges between vertices of S. Then, for any $B \subseteq E \setminus E(S)$, we have

$$\mathbb{E}_\mu[|B \cap T|] - q \le \mathbb{E}_\mu[|T \cap B| \mid T \text{ induces a tree in } S] \le \mathbb{E}_\mu[|B \cap T|]$$

   Hint: Recall that every spanning tree of G has exactly $|V| - 1$ edges.

9) **Submit by May 7th.** Let $G \sim G_{n,p}$ with $p = D/n$ and $D > 0$ is some fixed number. Let X be the number of isolated vertices in G (i.e., the number of vertices that have no neighbors). Determine $\mathbb{E}[X]$ and then show that for any $\lambda > 0$,

$$\mathbb{P}\left[|X - \mathbb{E}[X]| \ge 4\lambda\sqrt{Dn}\right] \le e^{-c\lambda^2},$$

   for some universal constant c.

10) **Submit by May 14th**.

- Show that the cover time of any unweighted d-regular graph $G = (V, E)$ is $O(n^2 \log n)$.
  Hint: What is the length of the shortest path between $u, v \in V$.
- We say an unweighted d-regular graph $G = (V, E)$ is a c-expander if for any $S \subseteq V$ where $|S| \leq |V|/2$, we have
  $$|E(S, \overline{S})| \geq cd|S|$$
  for some constant $c > 0$. Let G be a c-expander; show that for any $u, v \in V$,
  $$\text{Reff}(u, v) \leq \frac{1}{d} \cdot f(c)$$
  where $f(.)$ is a function of c independent of n, d. Consequently, argue that cover time of any c-expander graph is $O_c(n \log n)$ where $O_c(.)$ hides functions of c.
  Hint: Let p be the potential u-v flow. A potential cut is a cut $S_t = \{v : p_v \geq t\}$ for some threshold t. Also note that u has the highest potential and v has the lowest potential.

11) Let Z be a $d \times d$ random real, symmetric, PSD matrix such that $\text{rank}(\mathbb{E}[Z]) = d$. Suppose also that $Z \preceq L \cdot \mathbb{E}[Z]$ w.p. 1. for some $L \geq 1$. Show that if $Z_1, Z_2, \dots, Z_n$ are i.i.d. copies of Z, then for any $\epsilon > 0$, it holds that
$$\mathbb{P}\left[(1-\epsilon)\mathbb{E}[Z] \preceq \frac{1}{n}\sum_i Z_i \preceq (1+\epsilon)\mathbb{E}[Z]\right] \geq 1 - 2de^{-\epsilon^2 n/4L}.$$

12) The Ising model is defined as follows: We have (an undirected) graph $G = (V, E)$ and we have a particle at every state i with a "spin" $\sigma_i \in \{-1, +1\}$. The probability of a configuration $\sigma : V \to \{+1, -1\}$ is proportional to
$$\prod_{\{i,j\} \in E} e^{\beta \sigma_i \sigma_j}$$
We want to run the Heat-Bath Markov chain to generatre a random sample of these distributions: First, we sample a vertex i from V uniformly at random, then we "forget" σ_i, and instead, we sample σ_i from the stationary distribution conditioned on the state of every other vertex. Because of the locality of the weights (defined above) it is enough to sample the spin of the i-th vertex from the stationary distribution conditioned on σ_j for all neighbors vertices $j \sim i$.

Suppose we run this chain on a $\sqrt{n} \times \sqrt{n}$ torus. That is we assume that vertices in the last column are adjacent to vertices in the first column and similarly the vertices in the last row are adjacent to vertices in the first row. So, every vertex has degree exactly 4.

Use Path Coupling to show that there is a universal constant $c > 0$ (independent of n), such that for all $0 < \beta < c$, the chain mixes in time $O(n \log n)$.