12.1 Properties of Effective Resistance

We continue our discussion of effective by giving an overview of basic properties of effective resistances.

Lemma 12.1 (Metric Property). For any triple of vertices $s, t, u$,
\[ \Reff(s, t) + \Reff(t, u) \geq \Reff(s, u). \]

Proof. By (??),
\[
\Reff(s, u) = b_{s,u}^T L_G b_{s,u}
= (b_{s,t} + b_{t,u})^T L_G (b_{s,t} + b_{t,u})
= b_{s,t}^T L_G b_{s,t} + b_{t,u}^T L_G b_{t,u} + 2b_{s,t}^T L_G b_{t,u}
= \Reff(s, t) + \Reff(t, u) + 2b_{s,t}^T L_G b_{t,u}.
\]

So, we just need to show that the last term in the RHS is non-positive. The last term is equal to $p(t) - p(u)$ when we send one unit of flow from $s$ to $t$. But, this means that $t$ has the lowest potential in the network, so $p(u) \geq p(t)$ as required.

Lemma 12.2 (Rayleigh Monotonicity Property). For a weight function $w : E \to \mathbb{R}_+$ let $\Reff_w(., .)$ be the effective resistance function when the conductance of each edge $e \in E$ is $w(e)$. For any $w, w'$ such that $w \leq w'$ and any $s, t \in V$,
\[ \Reff_w(s, t) \geq \Reff_{w'}(s, t). \]

Proof. Let $x$ be the one unit electrical flow from $s$ to $t$ with respect to $w$. Since $w \leq w'$,
\[
\Reff_w(s, t) = \sum_{e \in E} \frac{x(e)^2}{w(e)} \geq \sum_{e \in E} \frac{x(e)^2}{w'(e)}.
\]

Since $x$ is a feasible flow that sends one unit of flow from $s$ to $t$, by Thompson’s law, the RHS is at least the energy of the electrical flow that sends one unit from $s$ to $t$ w.r.t. $w'$.

Lemma 12.3 (Convexity). The effective resistance is convex w.r.t. the conductances and is concave w.r.t. resistances. In particular, for any $s, t$,
\[
\frac{1}{2}(\Reff_{1/w_1}(s, t) + \Reff_{1/w_2}(s, t)) \leq \Reff_{1/(w_1 + w_2)}(s, t), \quad (12.1)
\]
\[
\frac{1}{2}(\Reff_{w_1}(s, t) + \Reff_{w_2}(s, t)) \geq \Reff_{(w_1 + w_2)/2}(s, t), \quad (12.2)
\]
12.1.1 Bounding the Effective Resistance

?? gives a simple method to upper bound the effective resistance of a pair of vertices: All we need to do is to find a flow from \( s \) to \( t \) of small \( \ell_2^2 \) energy. Conversely, the following lemma gives a lower-bound on the effective resistance of a pair of vertices.

**Lemma 12.4** (Nash Williams Inequality). Let \( S_1, S_2, \ldots, S_k \subseteq V \) such that for all \( 1 \leq i \leq k \), \( s \in S_i \) and \( t \notin S_i \). If for all \( 1 \leq i < j \leq k \), \( E(S_i, \overline{S_j}) \cap E(S_j, \overline{S_i}) = \emptyset \), then

\[
\text{Reff}(s, t) \geq \sum_{i=1}^{k} \frac{1}{\sum_{e \in E(S_i, \overline{S_i})} w(e)}.
\]

**Proof.** Suppose \( x \) sends one unit of flow from \( s \) to \( t \). We lower bound \( \mathcal{E}(x) \) with the expression in the RHS. Since the cuts corresponding to \( S_1, \ldots, S_k \) are disjoint, we can write

\[
\mathcal{E}(x) \geq \sum_{i=1}^{k} \sum_{e \in E(S_i, \overline{S_i})} \frac{x^2(e)}{w(e)}.
\]

Therefore, it is enough to show that for each \( 1 \leq i \leq k \),

\[
\sum_{e \in E(S_i, \overline{S_i})} \frac{x(e)^2}{w(e)} \geq \frac{1}{\sum_{e \in E(S_i, \overline{S_i})} w(e)}.
\] (12.3)

Since \((S_i, \overline{S_i})\) separates \( s, t \), \( \sum_{e \in E(S_i, \overline{S_i})} |x(e)| \geq 1 \). Therefore, by Cauchy-Schwarz inequality,

\[
1 \leq \left( \sum_{e \in E(S_i, \overline{S_i})} \frac{|x(e)|}{\sqrt{w(e)}} \cdot \sqrt{w(e)} \right)^2 \leq \left( \sum_{e \in E(S_i, \overline{S_i})} \frac{x(e)^2}{w(e)} \right) \cdot \left( \sum_{e \in E(S_i, \overline{S_i})} w(e) \right).
\]

This proves (12.3) and completes the proof of the lemma. \( \Box \)

As a simple application of the above lemma we can show that in a \( \sqrt{n} \times \sqrt{n} \) grid there is a pair of vertices \( s, t \) such that \( \text{Reff}(s, t) \geq \Omega(\log(n)) \).

Next, we discuss methods for upper bounding the effective resistance between a pair of vertices \( s, t \). Note that to upper bound the effective resistance it is enough to construct a feasible flow that sends one unit of flow from \( s \) to \( t \), then the energy of the flow will give an upper bound on \( \text{Reff}(s, t) \).

For example, suppose there are \( k \) edge disjoint paths each of length at most \( \ell \) from \( s \) to \( t \). Then we can construct \( x \) by sending \( 1/k \) amount of flow on each path and

\[
\mathcal{E}(x) \leq \sum_{i=1}^{k} \sum_{e \in P_i} x(e)^2 = \sum_{i=1}^{k} \sum_{e \in P_i} \frac{1}{k^2} \leq \frac{k \ell}{k^2}.
\]

Unfortunately, we may not be able to find many edge disjoint paths between \( s, t \) even though \( \text{Reff}(s, t) \) is small. For example, in a \( k \)-dimensional hypercube there are at most \( k \) edge disjoint paths between each pair of vertices because the degree of each vertex is \( k \). But because the length of each path between \( s = 00\ldots0 \) and \( t = 11\ldots1 \) is at least \( k \), the best upper bound that we can get is \( O(1) \).
12.2 Hitting Time

Consider a simple random walk on a (weighted) graph $G$, that is at a given vertex $u$ it goes to random neighbor $v$ with probability proportional to $w_{u,v}$, i.e.,

$$P[X_1 = v | X_0 = u] = \frac{w_{u,v}}{d_w(u)},$$

where $d_w(u) = \sum_{v' \sim u} w_{u,v'}$ is the weighted degree of $u$. In this section we study the hitting time: for two vertices $u, v \in V$, we define the hitting time $H_{u,v}$ from $u$ to $v$ as the expected number of steps for the random walk to hit $v$ (for the first time) when started at $u$. Formally, define the random variable

$$T = \min\{ t \geq 0 : X_t = v \}.$$

Then $H_{u,v} = E[T | X_0 = u]$. Note that his quantity is not necessarily symmetric, i.e., $H_{u,v} \neq H_{v,u}$ in general.

Lemma 12.5. Let $b$ be the vector which injects $d_w(u)$ unit at any vertex $u$ and extracts $d_w(V)$ at $v$, and let $p$ be the corresponding potential vector. For any vertex $u \neq v$,

$$H_{u,v} = p(u) - p(v).$$

Proof. Fix a vertex $v$, and let $h : V \to \mathbb{R}_{\geq 0}$ be the hitting time vector. Then, for any $u \neq v$, we can write the following system of linear equations for the Hitting time:

$$h(u) = 1 + \sum_{u'} \frac{w_{u,u'}}{d_w(u)} h(u').$$

Equivalently,

$$d_w(u) = \sum_{u'} w_{u,u'}(h(u) - h(u')) = L_G h(u)$$

Adding up the above equalities for $u \neq v$ we obtain,

$$d_w(V) - d_w(v) = \sum_{u'} w_{u',v}(-h(v) + h(u')),$$

i.e., the constraint for $v$ is redundant.

Now, if we consider the demand vector $b = d_w - 1_v d_w(V)$, i.e., we inject $d_w(u)$ at any vertex $u \in V$ and we extra all the flow at $v$ and solve for $p$ then $p$ exactly satisfies all of the above equations:

$$L_G p = b$$

As alluded to before these equations have a unique solution (when the graph is connected) up to a shift, i.e., if $h$ is a solution then so is $h + \alpha 1$ for any $\alpha \in \mathbb{R}$. So, we add one extra constraint that $h(v) = 0$. Equivalently, $p(u) - p(v) = h(v) - h(u) = h(v)$ as desired.

12.3 Commute Time

Given a pair of vertices $u, v$, the commute time $C_{u,v}$ is defined as:

$$C_{u,v} := H_{u,v} + H_{v,u}.$$
Theorem 12.6 (Commute Time). For any weighted graph $G = (V, E)$ and any pair of vertices $u, v$,

$$C_{u,v} = \text{Reff}(u,v)d_w(V).$$

Proof. We build on the proof of Theorem 12.6. We consider 3 different demand vectors:

1. Let $b^A$ be the demand vector that injects $d_w(x)$ at any vertex $x$ and extracts $d_w(V)$ at $v$.
2. Let $b^B$ be the demand vector that injects $d_w(V)$ at $u$ and extracts $d_w(x)$ at any vertex $x$.
3. Let $b^C = b^A + b^B$. In particular, $b^C$ injects $d_w(V)$ at $u$ and extracts it at $v$.

By Theorem 12.6,

$$H_{u,v} = p_{b^A}(u) - p_{b^A}(v).$$

By symmetry,

$$H_{v,u} = -(p_{b^B}(v) - p_{b^B}(u))$$

So,

$$C_{u,v} = p_{b^A}(u) + p_{b^B}(u) - (p_{b^A}(v) + p_{b^B}(v)) = p_{b^C}(u) - p_{b^C}(v).$$

Most importantly, in the second equality we used that $p$ can be obtained by solving a linear system involving the same matrix $L_G^\dagger$.

But by definition of effective resistance

$$p_{b^C}(u) - p_{b^C}(v) = d_w(V)(p_{b^C}(u) - p_{b^C}(v)) = d_w(V)\text{Reff}(u,v).$$

As an application, we can use ?? to give another proof of Lemma 12.1. Say for a triple of vertices $s, t, u$ we want to show

$$\text{Reff}(s,t) + \text{Reff}(t,u) \geq \text{Reff}(s,u)$$

By ?? this is equivalent to

$$C(s,t) + C(t,u) \geq C(s,u).$$

Now, by the definition of expected Hitting time this is equivalent to

$$H(s,t) + H(t,u) + H(u,t) \geq H(s,u) + H(u,s).$$

Rearranging the terms we need to show

$$(H(s,t) + H(t,u)) + (H(u,t) + H(t,s)) \geq H(s,u) + H(u,s).$$

But, for an triple of vertices $s, t, u$ we have

$$H(s,t) + H(t,u) \geq H(s,u).$$

The expected number of steps to go from $s$ to $u$ is smaller than the expected number of steps to go from $s$ to $u$ while visiting $t$ along the way.
12.4 Cover Time

The cover time of $G = (V, E, w)$ starting from $u$ is the quantity $cov_u(G)$ which is the expected number of steps needed to visit every vertex of $G$ started at $u$. Again, we can define this formally: Let

$$T = \min\{t \geq 0 : \{X_0, X_1, \ldots, X_t\} = V\}.$$ 

Then,

$$cov_u(G) = E[T|X_0 = u].$$

Finally, we define the cover time of $G$ as $cov(G) = \max_{u \in V} cov_u(G)$.

We can now use Theorem 12.6 to give a universal upper bound on the cover time of any graph.

**Theorem 12.7** (Cover Time). For any unweighted connected graph $G = (V, E, w)$, we have $cov(G) \leq 2|E|(|V| - 1)$.

**Proof.** Fix a spanning tree $T$ of $G$. Then we have

$$cov(G) \leq \sum_{(x, y) \in T} C_{x, y}.$$ 

The right-hand side can be interpreted as a very particular way of covering the graph $G$: First, given the tree $T$, we can turn it into a TSP tour by visiting every edge twice, we start at some node $x_0$ and "walk" around the edges of the $T$ in order $x_0, x_1, x_2, \ldots, x_{2(n-1)}$. In such a case the expected time from $x_0$ to get to $x_1$, is $H_{x_0, x_1}$, and similarly to cover $G$ it takes at most,

$$\sum_{i=0}^{2(n-1)} H_{x_i, x_{i+1}} = \sum_{(x, y) \in T} C_{x, y}.$$ 

Note that this is just one particular way to visit every node of $G$, so it gives an upper bound on the cover time. Finally, we note that if $\{x, y\}$ is an edge of the graph, then by Theorem 12.6 we have $C_{xy} = 2|E|\Reff(x, y) \leq 2|E|$. Here we use the fact that for every edge $\{x, y\}$ of $G$, the effective resistance is at most the resistance of the edge connecting $x$ to $y$. A candidate flow is just send one unit of flow directly from $x$ to $y$ on the edge. 

12.5 Examples of Cover Time

First observe that if $G$ is a $d$ regular graph by a direct application of Theorem 12.7,

$$cov(G) \leq dn(n - 1).$$

**The path** Consider first $G$ to be the path on vertices $\{0, 1, \ldots, n\}$. Then

$$H_{0, n} + H_{n, 0} = C_{0, n} = 2n\Reff(0, n) = 2n^2.$$ 

Since $H_{0, n} = H_{n, 0}$ by symmetry, we conclude that $H_{0, n} = n^2$. Note that Theorem 12.7 implies that $cov(G) \leq 2n^2$, and clearly $cov(G) \geq H_{0, n} = n^2$, so the upper bound is off by at most a factor of 2.
The lollipop. Consider next the "lollipop graph" which is a path of length \(n/2\) from \(u\) to \(v\) with an \(n/2\) clique attached to \(v\). We have

\[
H_{u,v} + H_{v,u} = C_{u,v} = \Theta(n^2)\text{Reff}(u,v) = \Theta(n^3).
\]

On the other hand, we have already seen that \(H_{u,v} = \Theta(n^2)\). We conclude that \(H_{v,u} = \Theta(n^3)\), hence \(cov(G) = \Omega(n^3)\). Again, the bound of Theorem 12.7 is \(cov(G) \leq O(n^3)\), so it’s tight up to a constant factor here as well.

The complete graph. Finally, consider the complete graph \(G\) on \(n\) nodes. In this case, Theorem 12.7 gives \(cov(G) \leq O(n^3)\) which is way off from the actual value \(cov(G) = \Theta(n \log n)\) (since this is just the coupon collector problem).

12.6 Mathews Bound

**Theorem 12.8 (Mathews Bound).** If \(G = (V, E)\) is an unweighted connected graph and \(R_{\text{max}} := \max_{x,y \in V} \text{Reff}(x,y)\) is the maximum pairwise effective resistance in \(G\), then

\[
|E|R_{\text{max}} \leq cov(G) \leq O(\log n)|E|R_{\text{max}}.
\]

**Proof.** One direction is straightforward:

\[
cov(G) \geq \max H_{u,v} \geq \frac{1}{2} \max_{u,v} C_{u,v} \geq \frac{1}{2}|E| \max \text{Reff}(u,v) = |E|R_{\text{max}}.
\]

For the other direction, we will examine a random walk of length \(2c|E|R_{\text{max}} \log n\) divided into \(\log n\) epochs of length \(2c|E|R_{\text{max}}\). Note that for any vertex \(v\) and any epoch \(i\), we have

\[
P[v \text{ unvisited in epoch } i] \leq \frac{1}{c}.
\]

This is because no matter what vertex is the first of epoch \(i\), we know that the hitting time to \(v\) is at most \(\max_u H_{u,v} \leq \max_u C_{u,v} = 2|E|R_{\text{max}}\). Now Markov’s inequality tells us that the probability it takes more than \(2c|E|R_{\text{max}}\) steps to hit \(v\) is at most \(1/c\). Therefore, the probability that we don’t visit \(v\) in any epoch is at most \(e^{-\log n} \leq n^{-\log c}\), and by a union bound, the probability that there is some vertex left unvisited after all the epochs is at most \(n^{1-\log c}\). We conclude that

\[
cov(G) \leq 2c|E|R_{\text{max}} \log n + n^{1-\log c}n^3
\]

where we have used the weak upper bound on the cover time provided by Theorem 12.7. Choosing \(c\) to be a large enough constant makes the second term negligible, yielding

\[
cov(G) \leq O(|E|R_{\text{max}} \log n),
\]

as desired.