CSE 525: Randomized Algorithms **Spring 2023**

Lecture 11: Effective Resistance and Simple Random Walks

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

In this lecture we overview the connection of the effective resistance and simple random walks in a graph.

11.1 Electrical Flows

The notion of *electrical flows* arises naturally when we treat our graph as a resistor network. Given a graph $G = (V, E)$ with weights $w(.)$ on the edges, we replace each edge e with a resistance of resistor $1/w(e)$. In other words, think of $w(e)$ as the conductance of the edge e. In this lecture we study how electricity flows in such a network.

Now, we write two underlying properties of electrical flows. The first one is the *flow conservation property.* Say we are sending one unit of flow from s to t. The flow conservation property says that for any vertex $v \neq s, t$, the sum of the flows into v is zero, this sum is $+1$ for s and -1 for t. For any edge $e = \{u, v\}$ we fix an arbitrary orientation say $u \to v$. We let $x(e)$ be the flow along edge e in that direction, i.e., $x(e)$ is non-negative if electricity is going from u to v and it is non-positive otherwise.

Let $\delta^-(v)$ be the neighbors u of v where the edge $\{u, v\}$ is oriented from u to v, and $\delta^+(v)$ be the rest of the neighbors of v. Then,

$$\sum_{u \in \delta^+(v)} x(e) - \sum_{u \in \delta^-(v)} x(e) = \begin{cases} +1 & \text{if } v = t \\ -1 & \text{if } v = s \\ 0 & \text{otherwise.} \end{cases}$$

We can rewrite the above equality as follows

$$B^\intercal x = b_{s,t}, \tag{11.1}$$

where $B \in \mathbb{R}^{E \times V}$ is the edge adjacency matrix, i.e., the $e = u \to v$-th row of B is $b_e = \mathbf{1}_u - \mathbf{1}_v$. Similarly, $b_{s,t} = \mathbf{1}_s - \mathbf{1}_t$.

The second property is the *Ohm's law.* This property implies that the electrical flows are potential flows. That is, if x is an electrical flow, then we can assign potentials $p : V \to \mathbb{R}$ such that for any edge $e = u \to v$,

$$x(e) = w(e) \cdot (p(u) - p(v)),$$

where w is the vector of conductances or inverse resistances. We use $W \in \mathbb{R}^{E \times E}$ to denote the diagonal matrix where for each edge e, $W(e, e) = w(e)$. We can rewrite the above equality as follows:

$$x = WBp. \tag{11.2}$$

Putting (11.1) and (11.2) together we get

$$B^\intercal WBp = L_G p = b_{s,t}. \tag{11.3}$$

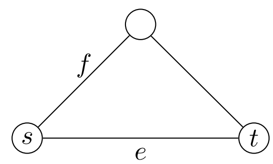

Figure 11.1: In this example if we send one unit of flow form s to t then 2/3 of the flow goes along the edge e and 1/3 goes through f, f'. Therefore, the potential difference between the endpoints of edges f is 1/3.

To see that first identity note that for any edge $u \to v$, $b_{u,v} b_{u,v}^T = L_e$ is the Lapacian of e, so $w_e b_{u,v} b_{u,v}^T$ is the weighted Laplacian of e. Summing over all edges of G we obtain the Laplacian of the graph L_G.

Note that L_G is in general not invertible since $L_G \mathbf{1} = 0$. But if G is connected we can write $L_G = \sum_{i=2}^n \lambda_i v_i v_i^T$ where $\lambda_i > 0$ and $\langle v_i, \mathbf{1} \rangle = 0$ for all $i \geq 2$. Then,

$$L_G^\dagger = \sum_{i=2}^n \frac{1}{\lambda_i} v_i v_i^T.$$

Since $\langle b_{s,t}, \mathbf{1} \rangle = 0$, there is a unique solution to the equation $L_G p = b_{s,t}$,

$$p = L_G^\dagger b_{s,t}. \tag{11.4}$$

The above argument naturally extends to any "demand" vector b (for the electrical flow) as long as $\langle b, \mathbf{1} \rangle = 0$.

11.2 Energy

Say a flow $y : E \to \mathbb{R}$ is feasible if it satisfies the flow conservation property, i.e., (11.1). The *energy* of a (feasible) flow y is defined as follows:

$$\mathcal{E}(y) = \sum_{e \in E} \frac{y(e)^2}{w(e)} = y^\intercal W^{-1} y, \tag{11.5}$$

Note that since W is a diagonal matrix, its inverse is simply the inverse of every element in the diagonal.

The energy can be seen as the ℓ_2^2 norm of a flow. Analogously, one can define the ℓ_r norm of a feasible flow y as follows:

$$\left(\sum_{e \in E} \frac{y(e)^r}{w(e)} \right)^{1/r}.$$

It turns out that among all feasible flows that send one unit of flow from s to t, the electrical flow is the one with the smallest energy (or the smallest ℓ_2 norm). To put this into perspective, in the maximum flow problem one is looking for a feasible flow from s to t that with the smallest ℓ_∞ norm.

Lemma 11.1 (Thompson's Law). *For any pair of vertices s, t, among all the flows that send one unit of flow from s to the t, the electrical flow has the smallest energy.*

Proof. Let x be the electrical flow. It follows by (11.2) and (11.4) that

$$x = W B L_G^\dagger b_{s,t}.$$

Therefore, the energy of x is equal to

$$\begin{aligned}\mathcal{E}(x) &= x^\intercal W^{-1} x \\ &= b_{s,t}^\intercal L_G^\dagger B^\intercal W W^{-1} W B L_G^\dagger b_{s,t} \\ &= b_{s,t}^\intercal L_G^\dagger b_{s,t}^\intercal. \end{aligned} \tag{11.6}$$

Now, let y be any feasible flow that sends one unit of flow from s to t, i.e., $B^\intercal y = b_{s,t}$. Therefore,

$$\mathcal{E}(x) = y^\intercal B L_G^\dagger B^\intercal y.$$

Now, all we need to show is that the above quantity is at most $y^\intercal W^{-1} y$. Say, $z = W^{-1/2} y$. We show

$$\mathcal{E}(x) = z^\intercal W^{1/2} B L_G^\dagger B^\intercal W^{1/2} z \le z^\intercal z = y^\intercal W^{-1} y.$$

To show that above inequality it is enough to show that

$$W^{1/2} B L_G^\dagger B^\intercal W^{1/2} \preceq I.$$

This is proved in Lemma 11.2 below. □

Lemma 11.2 (Spielman and Srivastava [**SS11**])**.** *The matrix* $Y = W^{1/2} B L_G^\dagger B^\intercal W^{1/2}$ *is a projection matrix, i.e.,* $Y^2 = Y$. *So,* $Y \preceq I$.

Proof.

$$\begin{aligned} YY^\intercal &= W^{1/2} B L_G^\dagger B^\intercal W^{1/2} W^{1/2} B L_G^\dagger B^\intercal W^{1/2} \\ &= W^{1/2} B L_G^\dagger L_G L_G^\dagger B^\intercal W^{1/2} \\ &= W^{1/2} B L_G^\dagger B^\intercal W^{1/2} = Y. \end{aligned}$$

Therefore all of the eigenvalues of Y are 0 or 1. □

11.3 Effective Resistance

The effective resistance between a pair of vertices s, t is defined as follows

$$\text{Reff}(s,t) = b_{s,t}^\intercal L_G^\dagger b_{s,t}. \tag{11.7}$$

By (11.4), $\text{Reff}(s,t)$ is the potential difference between s, t when we send one unit of electrical flow from s to t. Note that $\text{Reff}(s,t)$ is always non-negative because $L_G^\dagger$ is a PSD matrix. Equivalently, by (11.6), $\text{Reff}(s,t)$ is the energy of the electrical flow when we send one unit of flow from s to t.

The terminology of *effective resistance* originates from the following observation: If one removes all vertices of G except s, t and replaces the whole network with a resistance of resistor $\text{Reff}(s,t)$ between s, t, then, the energy (and the potential difference) of all electrical flows between s, t remains invariant.

The effective resistance of an edge $e = \{u, v\}$ is usually defined as the effective resistance between its endpoints. It turns out that if G is an unweighted graph, then the effective resistance of each edge is the probability that the edge is chosen in a uniform spanning tree distribution.

11.3.1 Properties of Effective Resistance

Lemma 11.3 (Metric Property)**.** *For any triple of vertices s, t, u,*

$$\text{Reff}(s,t) + \text{Reff}(t,u) \geq \text{Reff}(s,u).$$

Proof. By (11.7),

$$\begin{aligned}\text{Reff}(s,u) &= b_{s,u}^{\intercal} L_G^{\dagger} b_{s,u}\\ &= (b_{s,t} + b_{t,u})^{\intercal} L_G^{\dagger}(b_{s,t} + b_{t,u})\\ &= b_{s,t}^{\intercal} L_G^{\dagger} b_{s,t} + b_{t,u}^{\intercal} L_G^{\dagger} b_{t,u} + 2b_{s,t}^{\intercal} L_G^{\dagger} b_{t,u}\\ &= \text{Reff}(s,t) + \text{Reff}(t,u) + 2b_{s,t}^{\intercal} L_G^{\dagger} b_{t,u}.\end{aligned}$$

So, we just need to show that the last term in the RHS is non-positive. The last term is equal to $p(t) - p(u)$ when we send one unit of flow from s to t. But, this means that t has the lowest potential in the network, so $p(u) \geq p(t)$ as required. □

Lemma 11.4 (Rayleigh Monotonicity Property)**.** *For a weight function $w : E \to \mathbb{R}_+$ let $\text{Reff}_w(.,.)$ be the effective resistance function when the conductance of each edge $e \in E$ is $w(e)$. For any w, w' such that $w \leq w'$ and any $s, t \in V$,*

$$\text{Reff}_w(s,t) \geq \text{Reff}_{w'}(s,t).$$

Proof. Let x be the one unit electrical flow from s to t with respect to w. Since $w \leq w'$,

$$\text{Reff}_w(s,t) = \sum_{e \in E} \frac{x(e)^2}{w(e)} \geq \sum_{e \in E} \frac{x(e)^2}{w'(e)}.$$

Since x is a feasible flow that sends one unit of flow from s to t, by Lemma 11.1, the RHS is at least the energy of the electrical flow that sends one unit from s to t w.r.t. w'. □

Lemma 11.5 (Convexity)**.** *The effective resistance is convex w.r.t. the conductances and is concave w.r.t. resistances. In particular, for any s, t,*

$$\frac{1}{2}(\text{Reff}_{1/w_1}(s,t) + \text{Reff}_{1/w_2}(s,t)) \leq \text{Reff}_{2/(w_1+w_2)}(s,t). \tag{11.8}$$

$$\frac{1}{2}(\text{Reff}_{w_1}(s,t) + \text{Reff}_{w_2}(s,t)) \geq \text{Reff}_{(w_1+w_2)/2}(s,t), \tag{11.9}$$

11.3.2 Bounding the Effective Resistance

?? gives a simple method to upper bound the effective resistance of a pair of vertices: All we need to do is to find a flow from s to t of small ℓ_2^2 energy. Conversely, the following lemma gives a lower-bound on the effective resistance of a pair of vertices.

Lemma 11.6 (Nash Williams Inequality)**.** *Let $S_1, S_2, \ldots, S_k \subseteq V$ such that for all $1 \leq i \leq k$, $s \in S_i, t \notin S_i$. If for all $1 \leq i < j \leq k$, $E(S_i, \overline{S_i}) \cap E(S_j, \overline{S_j}) = \emptyset$, then*

$$\text{Reff}(s,t) \geq \sum_{i=1}^{k} \frac{1}{\sum_{e \in E(S_i, \overline{S_i})} w(e)}.$$

Proof. Suppose x sends one unit of flow from s to t. We lower bound $\mathcal{E}(x)$ with the expression in the RHS. Since the cuts corresponding to $S_1, \dots, S_k$ are disjoint, we can write

$$\mathcal{E}(x) \geq \sum_{i=1}^{k} \sum_{e \in E(S_i, \overline{S_i})} \frac{x^2(e)}{w(e)}.$$

Therefore, it is enough to show that for each $1 \leq i \leq k$,

$$\sum_{e \in E(S_i, \overline{S_i})} \frac{x(e)^2}{w(e)} \geq \frac{1}{\sum_{e \in E(S_i, \overline{S_i})} w(e)}. \tag{11.10}$$

Since $(S_i, \overline{S_i})$ separates s, t, $\sum_{e \in E(S_i, \overline{S_i})} |x(e)| \geq 1$. Therefore, by Cauchy-Schwarz inequality,

$$\begin{aligned} 1 &\leq \left(\sum_{e \in E(S_i, \overline{S_i})} \frac{|x(e)|}{\sqrt{w(e)}} \cdot \sqrt{w(e)} \right)^2 \\ &\leq \left(\sum_{e \in E(S_i, \overline{S_i})} \frac{x(e)^2}{w(e)} \right) \cdot \left(\sum_{e \in E(S_i, \overline{S_i})} w(e) \right). \end{aligned}$$

This proves (11.10) and completes the proof of the lemma. □

As a simple application of the above lemma we can show that in a $\sqrt{n} \times \sqrt{n}$ grid there is a pair of vertices s, t such that $\text{Reff}(s,t) \geq \Omega(\log(n))$.

Next, we discuss methods for upper bounding the effective resistance between a pair of vertices s, t. Note that to upper bound the effective resistance it is enough to construct a feasible flow that sends one unit of flow from s to t, then the energy of the flow will give an upper bound on $\text{Reff}(s,t)$.

Suppose there are k edge disjoint paths each of length at most ℓ from s to t. Then we can construct x by sending $1/k$ amount of flow on each path and

$$\mathcal{E}(x) \leq \sum_{i=1}^{k} \sum_{e \in P_i} x(e)^2 = \sum_{i=1}^{k} \sum_{e \in P_i} \frac{1}{k^2} \leq \frac{k\ell}{k^2}.$$

Unfortunately, we may not be able to find many edge disjoint paths between s, t even though $\text{Reff}(s,t)$ is small. For example, in a k-dimensional hypercube there are at most k edge disjoint paths between each pair of vertices because the degree of each vertex is k. But because the length of each path between $s = 00\dots0$ and $t = 11\dots1$ is at least k, the best upper bound that we can get is $O(1)$.

11.4 Hitting Time

Consider a simple random walk on a (weighted) graph G, that is at a given vertex u it goes to random neighbor v with probability proportional to $w_{u,v}$, i.e.,

$$\mathbb{P}[X_1 = v | X_0 = u] = \frac{w_{u,v}}{d_w(u)},$$

where $d_w(u) = \sum_{v' \sim u} w_{u,v'}$ is the weighted degree of u. In this section we study the hitting time: for two vertices $u, v \in V$, we define the hitting time $H_{u,v}$ from u to v as the expected number of steps for the random walk to hit v (for the first time) when started at u. Formally, define the random variable

$$T = \min\{t \geq 0 : X_t = v\}.$$

Then $H_{u,v} = \mathbb{E}[T | X_0 = u]$. Note that his quantity is not necessarily symmetric, i.e., $H_{u,v} \neq H_{v,u}$ in general.

Lemma 11.7. *Let b be the vector which injects $d_w(u)$ unit at any vertex u and extracts $d_w(V)$ at v, and let p be the corresponding potential vector. For any vertex $u \neq v$,*

$$H_{u,v} = p(u) - p(v).$$

Proof. Fix a vertex v, and let $h : V \to \mathbb{R}_{\geq} 0$ be the hitting time vector. Then, for any $u \neq v$, we can write the following system of linear equations for the Hitting time:

$$h(u) = 1 + \sum_{u'} \frac{w_{u,u'}}{d_w(u)} h(u')$$

Equivalently,

$$d_w(u) = \sum_{u'} w_{u,u'}(h(u) - h(u')) = L_G h(u)$$

Adding up the above equalities for $u \neq v$ we obtain,

$$d_w(V) - d_w(v) = \sum_{u'} w_{u',v}(-h(v) + h(u')),$$

i.e., the constraint for v is redundant.

Now, if we consider the demand vector $b = d_w - \mathbf{1}_v d_w(V)$, i.e., we inject $d_w(u)$ at any vertex $u \in V$ and we extra all the flow at v and solve for p then p exactly satisfies all of the above equations:

$$L_G p = b$$

As alluded to before these equations have a unique solution (when the graph is connected) up to a shift, i.e., if h is a solution then so is $h + \alpha \mathbf{1}$ for any $\alpha \in \mathbb{R}$. So, we add one extra constraint that $h(v) = 0$. Equivalently, $p(u) - p(v) = h(v) - h(u) = h(v)$ as desired.

□