Martingales

Today
- more martingales
  - Azuma-Hoeffding
- start online decision-making

A sequence of r.v.s $X_0, X_1, X_2, \ldots$ is called a discrete time martingale if the following conditions hold:

1. $E(|X_n|) < \infty$
2. $E(X_{n+1} | X_0, X_1, \ldots, X_n) = X_n$
3. $E(X_{n+1} - X_n | X_0, X_1, \ldots, X_n) = 0$

A sequence of r.v.s $X_0, X_1, \ldots$ is a martingale with respect to the sequence $Y_0, Y_1, \ldots$ if $Y_n \geq 0$ and the following conditions hold:

1. $X_n$ is a.f.m g $Y_0, Y_1, \ldots, Y_n$
2. $E(|X_n|) < \infty$
3. $E(X_{n+1} | Y_0, Y_1, \ldots, Y_n) = X_n$

Examples

1. Sums of independent random variables

   $Y_0 = 0, X_1, X_2, \ldots$ i.i.d. $E(Y_k) = 0 \quad \forall k$

   Define $X_n = Y_0 + X_1 + X_2 + \ldots + X_n$

   $\{X_n\}$ is a martingale w.r.t. $\{Y_k\}$
"Doob's" martingale process

$Y_1, Y_2, \ldots$ arbitrary seq of random vars
$X$ r.v. with finite expectation
$X_n = E(X | Y_1, \ldots, Y_n)$ forms martingale w.r.t. $\{Y_n\}
X_0 = E(X)$

Example: Edge exposure martingale

$G(n, \frac{1}{2})$ random graph
label $m=\binom{n}{2}$ potential edges $e_1, e_2, \ldots, e_m$

Let $f(G)$ be some function of the graph $f: 2^{\binom{n}{2}} \rightarrow \mathbb{R}$

$Y_j = \begin{cases} 1 & \text{edge } e_j \text{ present} \\ 0 & \text{otherwise} \end{cases}$  $Pr(Y_j = 1) = p$

$X_n = E[f(G) | Y_1, \ldots, Y_n]$
$X_0 = f(G) = E[f(G) | Y_1, \ldots, Y_m]$

Example: $f(G)$: sized max clique

$G(n, \frac{1}{2})$

$X_0 = E(\text{max clique})$
$X_1 = E(\text{max clique} | Y_1)$
$X_2 = E(\text{max clique} | Y_1, Y_2)$
$X_3 = E(\text{max clique} | Y_1, Y_2, Y_3)$
Some useful facts about martingales:

1. \( E(X_0) = E(X_n) \)
   
   by induction
   
   \[ E\left(\frac{E(X_{n+1} \mid Y_{0}, \ldots, Y_{n})}{Y_{0}, \ldots, Y_{n}}\right) = X_{n} \]
   
   \[ E\left(\frac{E(X_{n+1} \mid Y_{0}, \ldots, Y_{n})}{Y_{0}, \ldots, Y_{n}}\right) = E(X_{n}) \]
   
   \[ = E\left(\frac{E(X_{n+1})}{Y_{0}, \ldots, Y_{n}}\right) = \sum_{y} E\left(\frac{E(X_{n+1} \mid Y_{0}, \ldots, Y_{n})}{Y_{0}, \ldots, Y_{n}}\right) P_{Y_{0}, \ldots, Y_{n}}(Y_{0}, \ldots, Y_{n}) \]
   
   \[ = E(X) \]

2. Definition

   A r.v. \( T \) is called a "stopping time" wrt \( \{Y_{0}, Y_{1}, \ldots\} \)
   
   \( T \) takes values in \( \{0, 1, 2, \ldots\} \)
   
   and \( Y \), \( Y_{n} \geq 0 \), the event \( \{T=n\} \) is determined by \( Y_{0}, Y_{n} \)
   
   i.e. can determine \( Y \), \( T=n \) or \( T \neq n \) from knowledge \( Y \)
   
   values \( Y_{0}, \ldots, Y_{n} \)

   "know it when you see it"

   Examples
   
   - first time I win 5 games in a row
   - first time I win $150

   Non-example:
   
   - last time I win 5 games in a row

Optional Stopping Thm

\{Z_{i}^{j}\} is a martingale wrt \( \{X_{i}\} \)

For \( T \) a stopping time  "know it when you see it"

\[ E(Z_{T}) = E(Z_{0}) \]

whenever any of the following hold

- \( Z_{i}^{j} \) is bounded \( (\exists c \text{ s.t. } \forall i \mid Z_{i}^{j} \mid \leq c) \)
- \( T \) is bounded
- \( E(T) < \infty \) and \( \exists c \text{ s.t. } E\left[Z_{T} \mid X_{1}, X_{2}\right] \leq c \)
Tail inequalities

\[ E(X_n) = E(X_0) \quad \text{how far can it be from its expectation} \]

Azuma-Hoeffding Inequality

\[ X_0, \ldots, X_m \text{ martingale st. } \forall k \quad |X_k - X_{k-1}| \leq c_k \]

\[ c_k \text{ may depend on } k \]

Then \( \forall t > 0, \text{ any } R \geq 0 \]

\[ \Pr \left( |X_t - X_0| > R \right) \leq 2e^{-\frac{R^2}{2\sum c_k^2}} \]

Fact: \( X \) r.v., st. \( |X| \leq c \)

and \( E(X) = 0 \)

then \( E(e^{\lambda X}) \leq e^{\lambda^2 c^2 / 2} \)

Proof. By convexity \( f(x) = e^{\lambda x} \)

for any \( x \in [-c, c] \), we have

\[ e^{\lambda x} \leq (1 - \lambda c)e^{-\lambda c} + (1 + \lambda c)e^{\lambda c} = e(x) \]

If \( X \) with \( E(X) = 0 \) and \( |X| \leq c \)

then \( E(e^{\lambda X}) \leq E[e(X)] = \frac{e^{-\lambda c} + e^{\lambda c}}{2} = \frac{2}{2} \frac{2 \cdot (\frac{2c}{2})^k}{k!} \)

\[ \leq \sum_{k=0}^{\infty} \frac{(\frac{2c}{2})^k}{k!} = e \]

Corollary: \( E(X_{n+t} - X_t \| H_t) = 0 \) \( \Rightarrow \) \( X_{n+t} - X_t \leq \epsilon \Rightarrow \)

\[ E[e^{\lambda(X_{n+t} - X_t)} \| H_t] \leq e \]

\[ e^{\lambda^n c^2 / 2} \]
$$E[e^{x_{t+1}} | H_t] = \frac{\sum_{x_{t+1}} \left( e^{x_{t+1}} e^{x_t} \right) \Pr(x_{t+1} | H_t)}{\sum_{x_{t+1}} e^{x_t} \Pr(x_{t+1} | H_t)}$$

$$= e^{x_t} E[e^{x_{t+1}} | H_t]$$

$$\leq e^{x_t} e^{-\frac{\lambda^2}{2}}$$

$$\Rightarrow$$ taking expectations on both sides

$$E[e^{x_{t+1}}] \leq E[e^{x_t}] e^{-\frac{\lambda^2}{2}}$$

So by induction

$$E[e^{x_{t+1}}] \leq E[e^{x_0}] e^{-\frac{\lambda^2}{2}}$$

Finally,

$$\Pr(X_{t+1} > X_0) = \Pr(e^{x_{t+1}} > e^{x_0}) \leq e^{-\lambda R} E[e^{-\frac{\lambda^2}{2}}]$$

$$\leq e^{-\lambda R} e^{-\frac{\lambda^2}{2}}$$

Optimizing, we choose

$$\lambda = \frac{R}{\lambda^2}$$

$$\Rightarrow \Pr(X_{t+1} > X_0) \leq e^{-\frac{R^2}{2\lambda^2} \sum_{i=1}^{\infty} c_i^2}$$

Factory $2$ works for

$$\Pr(X_{t+1} < X_0)$$
Applications:

1. Balls in bins

\[ B_1, B_2, \ldots, B_n \] where \( B_i \) is bin for ball \( i \).

\( X \): \# empty bins.

\[ X_k = \mathbb{E}(X | B_1, \ldots, B_k) \]

\[ |X_k - X_{k-1}| \leq 1 \]

\[ \Pr \left( |X_n - X_0| > \sqrt{\frac{n}{3}} \right) \leq 2e^{-\frac{n}{2}} \]

\[ E(X) = n \left( 1 - \frac{1}{n} \right)^n \approx \frac{n}{e} \]

\[ X = \frac{n}{e} \pm O \left( \sqrt{\frac{n}{3}} \right) \]

2. Chromatic \# in random graph \( G(n, \frac{1}{2}) \)

Vertex exposure martingale

\[ X_k = \mathbb{E} \left[ \mathbb{I}(G) | u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_n \right] \]

\[ N(v) = \text{edges from } v_i \text{ to } v_i, \ldots, v_i \]

\[ X_0 = \mathbb{E} \left[ \mathbb{I}(G) \right] \]

\[ X_n = \mathbb{I}(G) \]

apply A-H

\[ \Pr \left( |\mathbb{I}(G) - \mathbb{E}(\mathbb{I}(G))| > R \right) \leq 2e^{-\frac{R^2}{2n}} \]

\[ R = w(\mathbb{I}(G)) \rightarrow 0 \]
Finding "interesting" patterns (e.g. in DNA seqs)

Let $X = (x_1, \ldots, x_n)$ be a sequence of characters chosen independently and uniformly at random from $\Sigma$, $|\Sigma| = s$

E.g. $\Sigma = \{A, T, C, G\}$

Let $B = (b_1, \ldots, b_k)$ be a fixed string of characters

Let $F$ r.v. be the number of occurrences of $B$ in string $X$

$F_i = \mathbb{E}(F | X_1, \ldots, X_i)$ Doblin martingale

$F_0 = \mathbb{E}(F)$

$F_n = F$

$\mathbb{E}(F) = (n^{-k+1}) \frac{1}{s^k}$

$|F_i - F_{i-1}| \leq k$ each char in at most $k$ matches

$\implies$ By Azuma-Hoeffding

$\mathbb{P}\left( \left| \frac{F_n - \mathbb{E}(F)}{F_0} \right| \geq \frac{2e}{\log n} \right) \leq 2e$

$\implies$ for $n = ckn$

$\mathbb{P}\left( \left| \frac{F_n - \mathbb{E}(F)}{ckn} \right| \geq k \right) \leq 2e$
One more cute (and useful) application of O.S.T.

Toss a fair coin

$X = \text{Exp } \# \text{ steps to see } HTH$?

$X = 2 + 2 + 1 + \frac{1}{2} X$

$\frac{X}{2} = 5 \Rightarrow X = 10$

$Y = \text{Exp } \# \text{ steps to see } HHH$?

$Y = 2 + 1 + \frac{1}{2} Y + \frac{1}{2} \left[ 1 + \frac{1}{2} Y \right]$

$y = \frac{3}{2} + \frac{3}{4} y \Rightarrow y = 14$. 
A martingale approach (for any pattern \( \sigma \))

**e.g.** \( \sigma = HTH \)

Sequence of independent coin tosses

\[ C_1, C_2, C_3, \ldots \quad C_i = \begin{cases} H & \text{up} \frac{1}{2} \\ T & \text{up} \frac{1}{2} \end{cases} \]

At each time step \( t \), a new gambler arrives and makes a series of double or nothing bets on \( \sigma \).

- **Bets**: $1
- **First toss comes up H**: loses $1
- **Wins gets $2**: bets $2 that next toss is T
  - **Loses $2**: wins $4
  - **Wins gets $4**: bets $4 that next toss is H
    - **Loses $4**: wins $8
    - **Wins gets $8**: loses $8

Let \( X_t \) represent the profit of all gamblers up to step \( t \).

\( \{ X_t \} \) is a martingale wrt \( \{ C_t \} \)

\[ E(X_{t+1} | C_1, C_2, C_3) = X_t \]

\[ X_t = \sum_{i=1}^{t} \text{profit of gambler that arrived at beginning of step } t \text{ and double or nothing bets on } \sigma \]

\( T \) be first time see \( \sigma = HTH \)

\[ E(X_T) = E(X_0) = 0 \]

\[ X_T = -T + 8 + 2 \]

\[ 0 = E(X_T) = -E(T) + 10 \]

\[ E(T) = 10 \]

\( \sigma = HHTHH \)

\[ E(T) = 8 + 4 + 2 = 14 \]

\[ X_T = -T + 8 + 4 + 2 \]

\[ E(T) = 14 \]
Repeated Online Decision Making and the Multiplicative Weights Algorithm

A set of possible actions $|A| = n$

$T$ a time horizon $A = \{1, 2, \ldots, n\}$

Setup:
At each time step $t = 1, \ldots, T$
- A decision maker picks an action $a_t \in A$
  where $\pi_t^a = \Pr(a_t = i)$
  $\pi^t = (\pi_t^1, \pi_t^2, \ldots, \pi_t^n)$
- An adversary picks reward vector
  $r^t = (r_t^1, r_t^2, \ldots, r_t^n)$
  where $r_t^i$ is the reward to Alg. if picked action $i$.
- Decision maker learns $r_t$

Goal of Alg.:

$$\text{maximize} \quad \sum_{t=1}^{T} \sum_{i=1}^{n} \frac{\pi_t^i r_t^i}{\pi_t^i r_t^i}$$

Examples:

1. Choosing a route
2. Choosing stocks to buy
Best possible world: \( \sum_{i=1}^{T} \max_{a \in A} r^+(i) \) \( \forall i \)

This benchmark way too strong:

Ex: \( A = \{1, 2\} \)
\( p^+(1) + p^+(2) = 1 \)
\( y_{p^+(1)} \geq 1/2 \Rightarrow r^+(1) = 1 \)
\( r^+(2) = 1 \)
\( \text{o.w.} \Rightarrow r^+(1) = 1 \)
\( r^+(2) = -1 \)
\( E[\text{reward of alg}] \leq 0 \)
\( (x) = 1 \)

\[
\text{Regret} (\hat{p}^1, \ldots, \hat{p}^T) = \frac{1}{T} \left[ \max_{a \in A} \sum_{t=1}^{T} r^+(a) - \sum_{t=1}^{T} \hat{p}^t r^+ \right]
\]

Goal: get \( \text{Reg} \rightarrow 0 \) as \( T \) gets large.

Obviously: "Follow the leader" up to \( t-1 \), let a be action with \( \max \sum_{t=1}^{T} r^+(a) \)

Use that on step \( t \)