Last time:

approx counting =\Rightarrow approx sampling

Q: given \mathcal{L} & distr \pi = (\pi_1, ..., \pi_n)

on altz \mathcal{L}

how to approx sample from this distr?

Cool idea:

- design a MC whose stationary distr
  is \pi
- show that it mixes in polynomial time
  i.e. after \( T = \text{poly}(n) \) steps
  \( \Pr(X_T = j) \approx \pi_j \) \( \forall j \in \mathcal{L} \)

Today

- another example of coupling
- martingales

Metropolis

- spectral approach
- coupling
Coupling

simple & elegant approach to bounding mixing time

Given an MC on \( \mathcal{X} \), a coupling is an MC on \( \mathcal{X} \times \mathcal{X} \) defining stochastic process \((X_t, Y_t)\) s.t.

1. each \( X_t \) & \( Y_t \) in isolation is faithful copy of MC
   \[ \Pr(X_{t+1} = x | X_t = x) = p_x \]
   \[ \Pr(Y_{t+1} = y | Y_t = y) = p_y \]

2. if \( X_t = Y_t \) then \( X_{t+1} = Y_{t+1} \)

Coupling Lemma

Let \((X_t, Y_t)\) be a coupling

Suppose \( \exists T \) s.t. \( \forall x,y \)

\[ \Pr(X_t \neq Y_t | X_0 = x, Y_0 = y) \leq \epsilon \]

Then \( \tau(\epsilon) \leq T \)

The \( \min \) \( + \) st. variation dist between dist over states of MC at time \( t \)

\[ \frac{1}{2} \sum_{x \in \mathcal{X}} |X_t(x) - Y_t(x)| = \max_{\alpha \in \mathcal{A}} |\Pr(A) - \Pr(A')| \]
Graph coloring

Input: \( G = (V, E) \) undirected, max degree \( \Delta \), \( k \)-colorable

Markov chain:
1. Pick vertex \( v \) & color c u.a.r.
2. Recolor \( v \) with \( c \) if legal

- aperiodic
- irreducible:

\[ \Delta \text{ neighbors of } v \]

\[ k > \Delta + 2 \]

\[ p_t > 0 \]

\[ p_{ss'} > 0 \]

\[ t > t^* \]

Conjecture: If \( k > \Delta + 2 \), this MC has poly mixing time.

(\text{Equation})
If $k > 4\Delta + 1$, then $MC$ has mixing time $O(n^{\Delta})$

Coupling: $X_t$ and $Y_t$ cross some $v \in C$ each step

$D_t = \{ v \mid X_t$ and $Y_t$ disagree on color of $v \}$

$A_t = V - D_t$

Here, $d_t$ may:

**Good moves ($d_t \downarrow$)**

Choose $v \in D_t$

**Bad moves ($d_t \uparrow$)**

Choose $v \notin D_t$

Before

After

All other moves cause no change in $d_t$

Prob $\frac{1}{2}$ each move $= \frac{1}{kn}$

$$E[d_{t+1} | d_t] = d_t + \frac{b_t - g_t}{kn} \leq d_t + \frac{2d_t \Delta - d_t (k - 2\Delta)}{kn}$$

$$= \frac{d_t (1 + \frac{4\Delta - k}{kn})}{d_t (1 - \frac{1}{kn})}$$

Since $k > 4\Delta + 1$

$$E[d_{t+1}] = E\left[ E[d_t | d_t] \right] \leq E[d_t] \left( 1 - \frac{1}{kn} \right)$$

$$E[d_t] \leq d_0 \left( 1 - \frac{1}{kn} \right)^t \leq d_0 e^{-\frac{tk}{n}} \leq \varepsilon$$

$t = kn \ln(n/k)$
\[ \Rightarrow \text{for } T = \ln \ln \left( \frac{N}{\varepsilon} \right) \]

\[ \Pr \left( X_T \neq Y_T \mid X_0 = x, Y_0 = y \right) \leq \varepsilon \Rightarrow \varepsilon(T) \leq \ln \ln \left( \frac{N}{\varepsilon} \right) \]

Bound can be improved to \( k \geq 2\Delta + 1 \) with more clever coupling see notes.
Martingales

A sequence of r.v.s \( X_0, X_1, \ldots \) is a martingale with respect to the sequence \( Y_0, Y_1, \ldots \) if \( Y_n \geq 0 \) and the following conditions hold:

- \( X_n \) is a.f.m. \( Y_0, Y_1, \ldots, Y_n \)
- \( E(\mid X_0) < \infty \)
- \( E(X_{n+1} \mid Y_0, Y_1, \ldots, Y_n) = X_n \)

Example: gambler plays sequence of fair games

- \( Y_i \): winnings on \( i \)-th game \( \quad E(Y_i) = 0 \)
- \( X_i \): gambler's total winnings at end of \( i \)-th game

\[
E(X_{i+1} \mid Y_0, Y_1, \ldots, Y_i) = E(X_i \mid Y_0, Y_1, \ldots, Y_i) + E(Y_{i+1} \mid Y_i, Y_{i-1}, \ldots, Y_0)
\]

\[
= E(X_i \mid Y_i, Y_{i-1}, \ldots) + E(Y_{i+1} \mid Y_i, Y_{i-1}, \ldots)
\]

\[= E(X_i \mid Y_i, Y_{i-1}, \ldots) + 0 \]

Martingale regardless of \( \text{amt bet each game} \), even \( Y_i \) that \( \text{amts} \) are dependent on previous results
Examples

1. Sums of independent random variables

\[ \sum_{i=0}^{n} Y_i = 0 \quad \text{iid} \quad \forall k \]

Define \( X_n = \sum_{i=0}^{n} Y_i \)

\( \{X_i\} \) is a martingale w.r.t. \( \{Y_i\} \)

\[
E(X_{n+1} | Y_{0:n}) = E(X_n + Y_{n+1} | Y_{0:n}) \\
= E(X_n | Y_{0:n}) + E(Y_{n+1} | Y_{0:n}) \\
= X_n + E(Y_{n+1}) \\
= X_n
\]

2. Variance of a sum

\[ \sum_{i=1}^{n} Y_i = 0 \quad \text{iid} \quad \forall k \quad E(Y_i^2) = \sigma^2 \]

Define \( X_n = (\sum_{i=1}^{n} Y_i)^2 - n\sigma^2 \)

\( \{X_i\} \) is a martingale w.r.t. \( \{Y_i\} \)

\[
E(X_{n+1} | Y_{0:n}) = E\left[ \left( \sum_{i=1}^{n+1} Y_i \right)^2 - (n+1)\sigma^2 \mid Y_{0:n} \right] \\
= E\left[ Y_{n+1}^2 + 2Y_{n+1}\left( \sum_{i=1}^{n} Y_i \right) + \left( \sum_{i=1}^{n} Y_i \right)^2 - (n+1)\sigma^2 \mid Y_{0:n} \right] \\
= E\left[ Y_{n+1}^2 + 2Y_{n+1}\left( \sum_{i=1}^{n} Y_i \right) + \left( \sum_{i=1}^{n} Y_i \right)^2 - (n+1)\sigma^2 \mid Y_{0:n} \right] \\
= \sigma^2 \left( X_n - \sigma^2 \right) \]

\( = X_n \)
"Dubois" martingale process

\[ Y_1, Y_2, \ldots \] arbitrary seq of random vars
\[ X \] r.v. with finite expectation
\[ X_n = E(X|Y_1, \ldots, Y_n) \] forms martingale wrt \{Y_n\}
\[ X_0 = E(X) \]

\[ E(X_{n+1}|Y_1, \ldots, Y_n) = E \left( E(X|Y_1, \ldots, Y_{n+1}) | Y_1, \ldots, Y_n \right) \]
\[ (\star) \; E(X|Y_1, \ldots, Y_n) = X_n \]

\[ E(V|W=w) = E \left[ E(V|U,w) | w \right] \]
prove this \(\uparrow\)
\[ \text{n.r. } \]
\[ E(V|W=w) \text{ w.p. } P(W=w) \]

Example: Edge exposure martingale

\( G(n, p) \) random graph
label \( m=\binom{n}{2} \) potential edges \( e_1, e_2, \ldots, e_m \)

Let \( f(G) \) be size of largest clique in \( G \)

\[ Y_j = \begin{cases} 1 & \text{edge } e_j \text{ present} \\ 0 & \text{otherwise} \end{cases} \quad \Pr(Y_j=1)=p \]

\[ X_j = E[f(G) | Y_1, \ldots, Y_j] \quad X_n = E[f(G)] \quad X_0 = f(G) \]

\[ E(X_j | Y_1, \ldots, Y_{j-1}) \]
Some useful facts about martingales:

1. \( E(X_n) = E(X_0) \)
   - by induction \( E(X_n|Y_0, \ldots, Y_{n-1}) = X_n \)
   - \( E[E(X_n|Y_0, \ldots, Y_{n-1})] = E(X_n) \)
   - \( = E(X_{n+1}) \)

2. Definition

   A r.v. \( T \) is called a "stopping time" wrt \( \{Y_t\} \) if
   - \( T \) takes values in \( \{0, 1, 2, \ldots\} \)
   - and \( Y_{T+n} > 0 \), the event \( \{T=n\} \) is determined by \( Y_0, Y_n \)
   - i.e. can determine \( Y_{T=n} \) or \( T\neq n \) from knowledge of \( Y_0, \ldots, Y_n \)
   - "know it when you see it"

Optional Stopping Theorem

\( \{Z_t\} \) is a martingale wrt \( \{X_t\} \)

For \( T \) a stopping time, "know it when you see it"

\( E(Z_T) = E(Z_0) \)

whenever any of the following hold
- \( Z_t \) is bounded \( (\exists c \text{ s.t. } |Z_t| \leq c) \)
- \( T \) is bounded
- \( E(T) < \infty \) and \( \exists c \text{ s.t. } E(E(Z_{T+n}|X_{11}, X_n) \leq c) \)
Applications of Optional Stopping Theorem

1) unbiased r.w. on line starting at 0

\[ -a \quad 0 \quad b \]

\[ Y_i = \begin{cases} 1 & \text{with prob } \frac{1}{2} \\ -1 & \text{with prob } \frac{1}{2} \end{cases} \]

\[ X_n = \sum_{i=1}^{n} Y_i \text{ martingale} \]

\[ T = \min \{ n \mid X_n = -a \text{ or } X_n = b \} \]

\( T \) is a stopping time

Let \( \nu_a = \text{Pr}(X_n \text{ reaches } -a \text{ before reaching } b) \)

By o.s.t. \( E(X_T) = E(X_0) = 0 \)

\[ E(X_T) = \nu_a (-a) + (1 - \nu_a) b = 0 \]

\[ \Rightarrow \nu_a = \frac{b}{a+b} \]

2) Same unbiased r.w. on line, same \( T \)

What is \( E(T) \)?

\[ Z_n = X_n - n \] is a martingale \([\text{variance of a sum } E(Y_i) = 1]\)

By o.s.t. \( E(Z_T) = E(Z_0) = 0 \)

\[ E(Z_T) = \left( \nu_a a^2 + (1 - \nu_a) b^2 \right) - E(T) = 0 \]

\[ E(T) = ab \]
Same questions: biased r.v.

\[ Y_i = \begin{cases} +1 & \text{p} \\ -1 & \text{q} \end{cases} \quad p > q \quad (=1-p) \]

\[ X_n = \frac{1}{n} \sum_{i=1}^{n} Y_i - n(q-p) \]

\[ X_n = \begin{pmatrix} q \end{pmatrix}^{\frac{1}{n}} Y_i ; \quad X_0 = 1 \]

\[ T = \min \left\{ n \mid \frac{1}{n} \sum_{i=1}^{n} Y_i = a \quad \text{or} \quad b \right\} \]

\[ v_a = \Pr \left( \frac{1}{n} \sum_{i=1}^{n} Y_i \text{ reaches } a \text{ before } b \right) \]

\[ E(X_T) = E(X_0) = 1 \]

\[ E(X_T) = v_a \left( \frac{q}{p} \right)^a + (1-v_a) \left( \frac{q}{p} \right)^b = 1 \]

\[ = \quad v_a = \frac{1 - \left( \frac{q}{p} \right)^b}{\left( \frac{q}{p} \right)^a - \left( \frac{q}{p} \right)^b} \]

\[ E(X_T) = v_a (-a) + (1-v_a) b - E(T) (q-p) = 0 \]

Does a fair coin repeatedly

\[ E(\# \text{tosses till see sequence HTTH}) ? \]