Today

- another example of coupling
- martingales

Last time:

approx counting $\Rightarrow$ approx sampling

Q: given Ω & distn $\pi = (\pi_1, \dots, \pi_{|\Omega|})$ over elts of Ω

how to approx sample from this distn?

Cool idea:

- design a MC whose stationary distn is π
- Show that it mixes in polynomial time

i.e. after $T = \text{poly}(n)$ steps

$$\Pr(X_T = j) \approx \pi_j \quad \forall j \in \Omega$$

Metropolis.

- spectral approach
- coupling

$\boxed{\text{Coupling}}$ simple & elegant approach to bounding mixing time

Given a MC P on Ω, a coupling is a MC on $\Omega \times \Omega$ defining stochastic process (X_t, Y_t) s.t.

① each of X_t & Y_t in isolation is faithful copy of MC

$\Pr(X_{t+1} = z \mid X_t = x) = P_{xz}$

$\Pr(Y_{t+1} = w \mid X_t = y) = P_{yw}$

② If $X_t = Y_t$ then $X_{t+1} = Y_{t+1}$

$\boxed{\text{Coupling Lemma}}$ Let (X_t, Y_t) be a coupling

Suppose $\exists\, T$ s.t. $\forall x, y$

$$\Pr(X_T \neq Y_T \mid X_0 = x, Y_0 = y) \le \varepsilon$$

Then $\tau(\varepsilon) \le T$

min t s.t. variation dist between dist over states of MC at time t X_t & π $\le \varepsilon$

$$\|D_1 - D_2\|_{TV} = \frac{1}{2} \sum_{x \in \Omega} |D_1(x) - D_2(x)| = \max_{A \subseteq \Omega} |D_1(A) - D_2(A)|$$

Graph coloring

Input: $G=(V,E)$ undirected; max degree Δ. k colors

Recall: if $k \geq \Delta+1$, k-colorable

Markov chain:

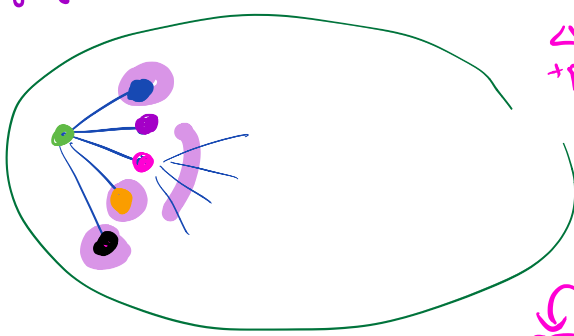
① pick vertex v & color c u.a.r.

② recolor v with c if legal

$k > \Delta+2$

- aperiodic
- irreducible:

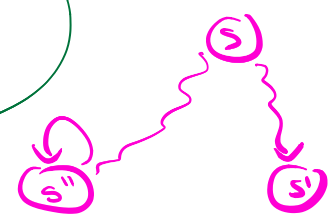
Δ neighbors + itself $\Delta+1$



- π uniform

P symmetric

$$\overset{v,c}{\underset{\longleftarrow}{\longrightarrow}}$$



$$P^t > 0$$

$$P^t_{ss'} > 0 \quad \forall t \geq t^*$$

Conjecture: $\forall$ $k \geq \Delta+2$, this MC has poly mixing time
even $O(n\log n)$

Thm If $k \geq 4\Delta+1$, then MC has mixing time $O(n\log n)$

Coupling: X_t & Y_t choose same v & c each step

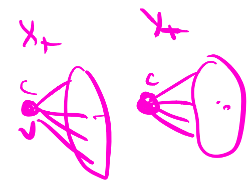
$D_t = \{ v \mid X_t \text{ \& } Y_t \text{ disagree on color of } v \}$ $\quad |D_t| = d_t$

$A_t = V - D_t$

here d_t may $\uparrow$

$\boxed{\text{Good moves } d_t \downarrow}$ $\quad g_t = \#\text{ good moves} \geq d_t(k-2\Delta)$

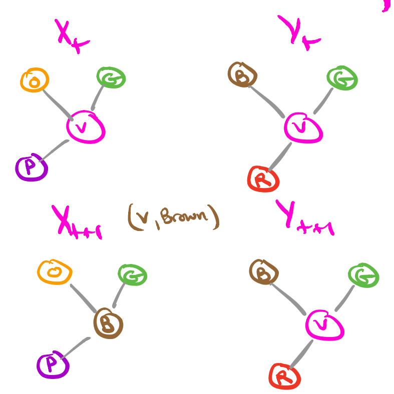
choose $v \in D_t$



$\boxed{\text{Bad moves } d_t \uparrow}$ $\quad b_t = \#\text{ bad moves} \leq \underset{\text{2 colors of } v \in D_t}{2}\, \underbrace{d_t \Delta}_{\#\text{ nodes that are neighbors of } v \in D_t}$

choose $v \notin D_t$
recolor in one chain
but not other.

c must be color of some neighbor of v in one chain but not other.
v neighbor of vertex in D_t



$E(X \mid Y) \to f(Y)$

$E(X|Y=y)$ w.p. $Pr(Y=y)$

$E(E(X|Y))$

$= \sum_y E(X|Y=y)\, Pr(Y=y)$

$= \sum_y \sum_x x\, Pr(X=x|Y=y)\, Pr(Y=y)$

$= \sum_x x \sum_y Pr(X=x \cap Y=y)$

$= \sum_x x\, Pr(X=x)$

$= E(X)$

$\boxed{E(E(X|Y)) = E(X)}$

All other moves cause no change in d_t

Prob of each move $= \frac{1}{kn}$

$\Rightarrow E[d_{t+1} \mid d_t] = d_t + \frac{b_t - g_t}{kn} \leq d_t + \frac{2d_t\Delta - d_t(k-2\Delta)}{kn}$

$= d_t\left(1 + \frac{4\Delta - k}{kn}\right) \leq d_t\left(1 - \frac{1}{kn}\right)$

since $k \geq 4\Delta+1$

$\Rightarrow E[d_t] = E[E[d_t|d_{t-1}]] \leq E[d_{t-1}]\left(1 - \frac{1}{kn}\right)$

$\Rightarrow E[d_t] \leq d_0\left(1-\frac{1}{kn}\right)^t \leq \underbrace{d_0}_{\leq n} e^{-t/kn} \leq \varepsilon \qquad t = kn\ln\left(\frac{n}{\varepsilon}\right)$

Markovs ineqn

$\Rightarrow$ for $T = kn\ln\left(\frac{n}{\varepsilon}\right)$

$$\Pr\big(\underbrace{X_T \neq Y_T}_{d_T \geq 1} \mid X_0 = x, Y_0 = y\big) \leq \varepsilon \quad \Rightarrow \tau(\varepsilon) \leq kn\ln\left(\frac{n}{\varepsilon}\right)$$

bound can be improved to $k \geq 2\Delta + 1$ with more clever coupling
see notes.

Martingales

A sequence of r.v.s $X_0, X_1, \dots,$ is a martingale with respect to the sequence $Y_0, Y_1, \dots$ if $\forall n \geq 0$ the following conditions hold:

- X_n is a fn of $\underline{Y_0, Y_1, \dots, Y_n}$ think of $Y_0, \dots Y_n$ history up to time n
- $E(|X_n|) < \infty$
- $E(X_{n+1} \mid Y_0, \dots, Y_n) = X_n$

Sequence of r.v.s $X_0, X_1, X_2, \dots$ called a martingale when it is a martingale wrt itself

- $E(|X_n|) < \infty$
- $E(X_{n+1} \mid X_0, X_1, \dots, X_n) = X_n$

$E(X_{i+1} \mid \underline{Y_1 \dots Y_n})$
r.v.

$E(X_{i+1} \mid Y_1 \dots Y_i)$
$= \underbrace{E(X_i \mid Y_1 \dots Y_i)}_{X_i} + \underbrace{E(Y_{i+1} \mid Y_1 \dots Y_i)}_{0}$

Example: gambler plays sequence of fair games

Y_i: winnings on i^{th} game $\quad E(Y_i) = 0$

X_i = gamblers total winnings at end of i^{th} game

$E(X_{i+1} \mid Y_1, \dots, Y_n) = X_i + E(Y_{n+1}) = X_i$

martingale regardless of amt bet each game, even if these amts are dependent on previous results

Examples

① Sums of indep random variables $\qquad E(Y_k \mid Y_1, \ldots, Y_{k-1}) = 0$

$Y_0 = 0 \qquad Y_1, \ldots, Y_n$ iid w/ $E(Y_k) = 0 \quad \forall k$

Define $\qquad X_n = Y_0 + Y_1 + Y_2 + \cdots + Y_n$

$\{X_t\}$ is a martingale w.r.t. $\{Y_t\}$

$$E(X_{n+1} \mid Y_0, \ldots, Y_n) = E(X_n + Y_{n+1} \mid Y_0, \ldots, Y_n)$$
$$= E(X_n \mid Y_0, \ldots, Y_n) + E(Y_{n+1} \mid Y_0, \ldots, Y_n)$$
$$= X_n + E(Y_{n+1})$$
$$= X_n$$

② Variance of a sum

$Y_1, \ldots, Y_n$ iid w/ $E(Y_k) = 0 \quad \forall k \qquad E(Y_k^2) = \sigma^2$

Define $\quad X_0 = 0 \qquad X_n = \left(\sum_{k=1}^{n} Y_k\right)^2 - n\sigma^2$

$\{X_t\}$ is a martingale wrt $\{Y_t\}$

$$E(X_{n+1} \mid Y_0, \ldots, Y_n) = E\Big[\overbrace{\Big(\sum_{k=1}^{n+1} Y_k\Big)^2 - (n+1)\sigma^2}^{X_{n+1}} \;\Big|\; Y_0, \ldots, Y_n\Big]$$

$$= E\Big[\underbrace{Y_{n+1}^2}_{\sigma^2} + \underbrace{\underbrace{2Y_{n+1}\Big(\sum_{k=1}^{n} Y_k\Big)}_{\text{Exp } 0}}_{0} + \underbrace{\Big(\sum_{k=1}^{n} Y_k\Big)^2 - (n+1)\sigma^2}_{X_n - \sigma^2} \;\Big|\; Y_0, \ldots, Y_n\Big]$$

$$= X_n$$

(3) "Doob's" martingale process

$Y_1, Y_2, \ldots$ arbitrary seq of random vars

X r.v. with finite expectation

$X_n = E(X \mid Y_1, \ldots, Y_n)$ forms martingale wrt $\{Y_n\}$

$X_0 = E(X)$

$$E(X_{n+1} \mid Y_1, \ldots, Y_n)$$
$$= E\big(E(X \mid Y_1, \ldots, Y_{n+1}) \mid Y_1, \ldots, Y_n\big)$$
$$(*) = E(X \mid Y_1, \ldots, Y_n) = X_n$$

$(*)$ $E(V \mid W) = E\big[E(V \mid U, W) \mid W\big]$

prove this ↑

r.v.

$E(V \mid W = w)$ w.p. $P(W = w)$

Example: Edge exposure martingale

$G(n,p)$ random graph

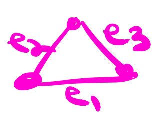
label $m = \binom{n}{2}$ potential edges $e_1, e_2, \ldots, e_m$

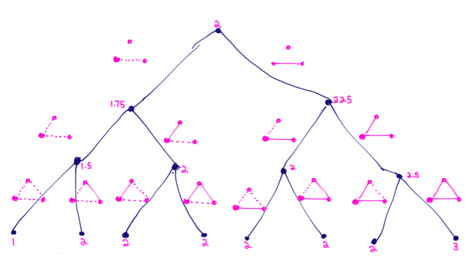
Let $f(G)$ be size of largest clique in G

$Y_j = \begin{cases} 1 & \text{if edge } e_j \text{ present} \\ 0 & \text{otherwise} \end{cases}$ $\quad Pr(Y_j = 1) = p$

$X_k = E[f(G) \mid Y_1, \ldots, Y_k]$ $\quad X_0 = E[f(G)]$

$X_m = f(G) = E(f(G) \mid Y_1, \ldots, Y_m)$





$G(n, \frac{1}{2})$

$n = 3$

$p = \frac{1}{2}$

$E(X_3 \mid Y_1 = 0, Y_2 = 1)$

$X_0 = E(\text{max clique})$

$X_1 = E(\text{max clique} \mid Y_1)$

$X_2 = E(\text{max clique} \mid Y_1, Y_2)$

$X_3 = E(\text{max clique} \mid Y_1, Y_2, Y_3)$

e_1 in or not

$E(X_3 \mid Y_1, Y_2) = X_2$

Some useful facts about martingales:

1. $E(X_n) = E(X_0)$

by induction $\quad E(X_{n+1} \mid Y_0, \dots, Y_n) = X_n$

$$\underbrace{E\left[E(X_{n+1} \mid Y_0, \dots, Y_n)\right]}_{= E(X_{n+1})} = E(X_n)$$

$E(X|Y=y)$ w.p. $Pr(Y=y)$

$$E(\overbrace{E(X|Y)}) = \sum_y E(X|Y=y) Pr(Y=y) = E(X)$$

2. Definition

$Y_0 \; Y_1 \; Y_2 \; \dots \; Y_n \; \dots$

A r.v. T is called a "stopping time" wrt $\{Y_t\}$ if

T takes values in $\{0,1,2,\dots\}$

and if $\forall n \geq 0$, the event $\{T=n\}$ is determined by $Y_0, \dots, Y_n$

i.e. can determine if $T=n$ or $T \neq n$ from knowledge of values $Y_0, \dots, Y_n$

"know it when you see it"

T: first time I win 5 games in a row ✓

: first I win $\geq$ \$100

~~last time I win 5 games in a row~~ ✗

$\boxed{\text{Optional Stopping Thm}}$

$\{Z_t\}$ is a martingale wrt $\{X_t\}$

For T a stopping time "know it when you see it"

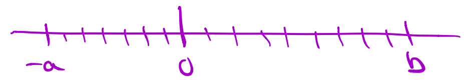
$$E(Z_T) = E(Z_0)$$

whenever any of the following hold

- Z_i's bounded $\quad (\exists c$ s.t. $\forall i \; |Z_i| \leq c)$
- T is bounded
- $E(T) < \infty$ and $\exists c$ s.t. $E\left(|Z_{i+1} - Z_i| \,\middle|\, X_1, \dots, X_i\right) \leq c$

Applications of Optional Stopping Theorem

(1) unbiased r.w. on line starting at 0



$Y_i = \begin{cases} 1 & \text{with prob } \frac{1}{2} \\ -1 & \text{" \quad "} \quad \frac{1}{2} \end{cases}$ $\qquad X_n = \sum_{i=1}^{n} Y_i$ martingale

$T = \min \{ n \mid X_n = -a \text{ or } X_n = b \}$

T is a stopping time

Let $v_a = Pr(X_n \text{ reaches } -a \text{ before reaching } b)$

By O.S.T. $\qquad \underline{E(X_T) = E(X_0) = 0}$

$E(X_T) = v_a \cdot (-a) + (1 - v_a) \cdot b = 0$

$\Rightarrow v_a = \frac{b}{a+b}$

(2) Same unbiased r.w. on line, same T

What is $E(T)$?

$Z_n = X_n^2 - n$ is a martingale [variance of a sum $E(Y_i^2) = 1$]

By O.S.T. $\qquad E(Z_T) = E(Z_0) = 0$

$E(Z_T) = \left(v_a a^2 + (1 - v_a) b^2 \right) - E(T) = 0$

$E(T) = ab$

Same questions: biased r.w.

$X_n = \underbrace{\sum_{i=1}^{n} Y_i - nE(Y_i)}$

$$Y_i = \begin{cases} +1 & p \\ -1 & q \end{cases} \qquad p > q \ (= 1-p)$$

$$\left.\begin{array}{l} X_n = \sum_{i=1}^{n} Y_i - n(p-q) \\ X_n' = \left(\frac{q}{p}\right)^{\sum_{i=1}^{n} Y_i} \qquad X_0' = 1 \end{array}\right\} \text{ martingales wrt } \{Y_n\}$$

check it!

$$T = \min\left\{ n \,\middle|\, \sum_{i=1}^{n} Y_i = -a \text{ or } = b \right\}$$

$$v_a = \Pr\left(\sum_{i=1}^{n} Y_i \text{ reaches } -a \text{ before } b\right)$$

$$E(X_T') = E(X_0') = 1$$

$$E(X_T') = v_a \left(\frac{q}{p}\right)^{-a} + (1-v_a)\left(\frac{q}{p}\right)^{b} = 1$$

$$\Rightarrow v_a = \frac{1 - \left(\frac{q}{p}\right)^{b}}{\left(\frac{q}{p}\right)^{-a} - \left(\frac{q}{p}\right)^{b}}$$

$$E(X_T) = v_a(-a) + (1-v_a)b - E(T)(p-q) = 0$$

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toss a fair coin repeatedly

E(# tosses till see sequence HTH) ?