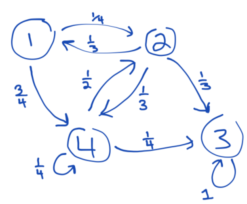
Finite Markov Chains

- random walk on directed graph
- each vertex is a "state" of MC.
- each arc describes corresponding transition probability



$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \end{array}$$

Use X_t to denote state at time t

$$\Pr(X_{t+1}=j \mid X_t = i) = p_{ij}$$

$P = (p_{ij})$ transition prob matrix

$\vec{p}^{\,t} = (p_1^t, p_2^t, \ldots, p_n^t)$ describes prob distn over states at time t

$$p_i^t = \Pr(X_t = i)$$

$\vec{p}^{\,0} = (1, 0, \ldots 0)$ means start in state i

$\vec{p}^{\,0} = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ means start in uniformly random state

$$\vec{p}^{\,t+1} = \vec{p}^{\,t} P \quad \Rightarrow \quad \vec{p}^{\,t+m} = \vec{p}^{\,t} P^m$$

called a Markov chain because it has "Markov property"
= next state depends on current state but not on history

Irreducible Markov chain

corresponding graph strongly connected

<u>period</u> of state i $= \gcd \{ n \geq 1 \mid P_{ii}^n > 0 \}$

Markov chain is <u>aperiodic</u> if period of every state is 1

All Markov chains we will consider will be finite, irreducible & aperiodic

$$\Rightarrow \quad \exists N > 0 \quad \text{s.t. } P^n \text{ strictly positive } \forall n \geq N$$

A $\boxed{\text{stationary}}$ distn $\vec{\pi}$ of a M.C. is a prob distn s.t. $\underbrace{\vec{\pi} = \vec{\pi} P}_{\forall j \quad \pi_j = \sum_i \pi_i p_{ij}}$ "fixed point"

$\Rightarrow \vec{\pi} = \vec{\pi} P^t$

$\boxed{\text{Fundamental Thm of Markov Chains}}$

For any finite, irreducible, aperiodic MC

1. $\exists$ stationary distn $\vec{\pi}$ (with $\pi_i > 0 \ \forall i$)
2. $\vec{\pi}$ is unique
3. $\pi_i = \frac{1}{h_{i,i}}$
4. $\forall i, j \quad \lim_{n \to \infty} P_{ij}^t = \pi_j$

Notation

$H_{ij} = \min_{t \geq 1} \{ X_t = j \mid X_0 = i \}$

$h_{ij} = E(H_{ij})$

h_{ii} = expected first return time.

Random walks on graphs.

Some interesting questions:

1) What is limiting dist'n of random walk? (i.e. stationary dist'n)
2) How long does it take before the walk approaches the limiting dist'n?
3) Starting from vertex s, what is the exp # of steps to first reach t?
4) How long does it take to reach every vertex at least once?

Proof of thm for d-regular graphs using spectral approach

$\pi = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$

$\boxed{\text{Spectral Thm}}$

If $M \in \mathbb{R}^{n\times n}$ symmetric, then

- all its eigenvalues are real
- $\exists$ full basis of orthonormal eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ where $\vec{v}_i \cdot \vec{v}_j = 1_{i=j}$

  $M\vec{v}_i = \lambda \vec{v}_i$

$$M = \sum_{i=1}^{n} \lambda_i v_i v_i^T = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} - & v_1^T & - \\ - & v_2^T & - \\ & \vdots & \\ - & v_n^T & - \end{pmatrix}$$

$\Phi^T \quad \Lambda \quad \Phi$

$\Phi^T \Phi = \Phi \Phi^T = I$

Monte Carlo Methods

collection of tools for estimating values thru sampling & estimation

(ε, δ) Approximation

A randomized alg gives an (ε, δ) approx for value V if the output X of the alg satisfies

$$\Pr(|X-V| > \varepsilon|V|) \le \delta$$

Example

Sample indep random vars whose mean is quantity we want to estimate

Let $X_1, X_2, \ldots, X_m$ iid. Bernoulli with $E(X_i) = \mu$

If $m \ge \frac{3 \ln(\frac{2}{\delta})}{\varepsilon^2 \mu}$ then

$$\Pr\left(\left| \frac{1}{m} \sum_{i=1}^{m} X_i - \mu \right| \ge \varepsilon\mu \right) \le \delta$$

Pf: Chernoff bounds

DNF Counting

Suppose want to know # satisfying assignments

$$(\bar{x}_1 \wedge x_2) \vee (x_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge \bar{x}_3 \wedge x_4) \vee (x_3 \wedge \bar{x}_4)$$