Finite Markov Chains & Random walks

Today
- intro Markov chains
- Algorithms
  - 2-SAT
  - bipartite matching in d-regular graph
- random walks on graphs

2-SAT Algorithm

- start w/ initial assignment
- Repeat up to $100n^2$ times until all clauses are satisfied
  - choose an arbitrary clause that is not satisfied
  - pick a variable in that clause at random & switch its value
- Repeat satisfying assignment if found; else return "unsatisfiable"

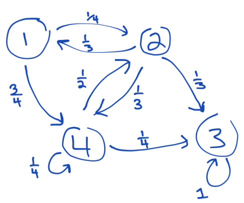
How to analyze?

Fix satisfying assignment S

Think of alg as random walk on line

Finite Markov Chains

- random walk on directed graph
- each vertex is a "state" of MC.
- each arc describes corresponding transition probability



$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array} \quad \begin{pmatrix} 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Use X_t to denote state at time t

$$\Pr(X_{t+1}=j \mid X_t = i) = p_{ij}$$

$P = (p_{ij})$ transition prob matrix

$\vec{p}^{\,t} = (p_1^t, p_2^t, \ldots, p_n^t)$ describes prob distn over states at time t

$$p_i^t = \Pr(X_t = i)$$

$\vec{p}^{\,0} = (1, 0, \ldots 0)$ means start in state i

$\vec{p}^{\,0} = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ means start in uniformly random state

Irreducible Markov chain

corresponding graph strongly connected

period of state i $= \gcd\{n \geq 1 \mid p_{ii}^n > 0\}$

Markov chain is **aperiodic** if period of every state is 1

All Markov chains we will consider will be finite, irreducible & aperiodic

$\Rightarrow \exists N > 0$ s.t. P^n strictly positive $\forall n \geq N$

A $\boxed{\text{stationary}}$ distn $\vec{\pi}$ of a M.C. is a prob distn s.t. $\underbrace{\vec{\pi} = \vec{\pi} P}$ "fixed point"

$\forall j \quad \pi_j = \sum_i \pi_i p_{ij}$

$\boxed{\text{Fundamental Thm of Markov Chains}}$

For any finite, irreducible, aperiodic MC

1. $\exists$ stationary distn $\vec{\pi}$ (with $\pi_i > 0 \ \forall i$)
2. $\vec{\pi}$ is unique
3. $\pi_i = \frac{1}{h_{i,i}}$
4. $\forall i,j \quad \lim_{n\to\infty} P^t_{ij} = \pi_j$

Notation

$T_{ij} = \min_{t \geq 1} \{X_t = j \mid X_0 = i\}$

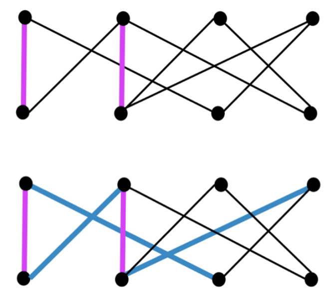
$h_{ij} = E(H_{ij})$

h_{ii} = expected first return time.

Maximum Matching in Regular Bipartite Graphs

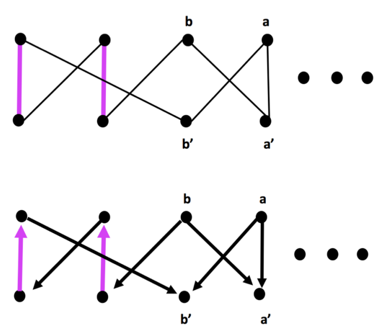
By Hall's Marriage Thm, regular bipartite graphs always have perfect matching

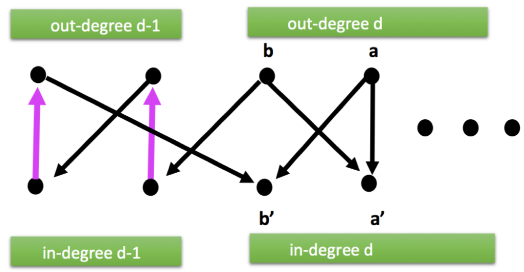
Traditional approach: augmenting path alg
repeatedly find one: can be done in $O(m)$ steps
using BFS

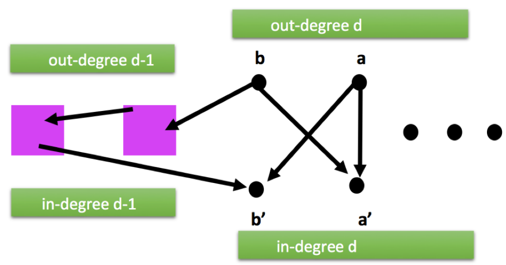


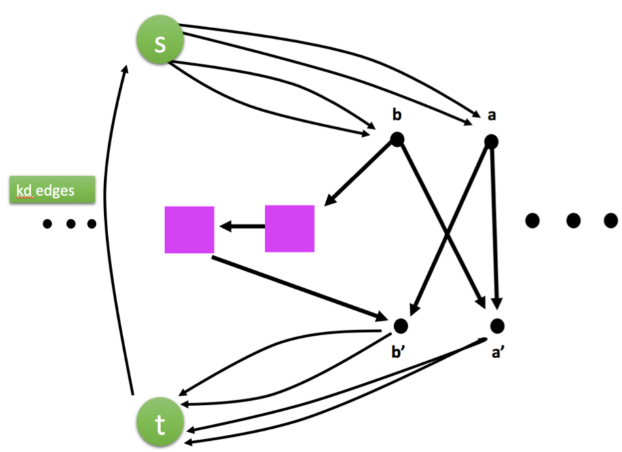
Prior to this best alg $O(m)$

Random walk based alg




out-degree d-1
out-degree d
b
a
b'
a'
in-degree d-1
in-degree d


out-degree d
out-degree d-1
b
a
b'
a'
in-degree d-1
in-degree d


s
b
a
kd edges
b'
a'
t

Random walks on graphs

$G = (V, E)$ undirected graph.

Some interesting questions:

1) What is limiting dist'n of random walk?
2) How long does it take before the walk approaches the limiting dist'n?
3) Starting from vertex s, what is the exp # of steps to first reach t?
4) How long does it take to reach every vertex at least once?