2nd Moment Method

Today

- Lovasz Local Lemma

- Chebychev's Inequality

  $\forall \lambda > 0 \quad Pr(|X-\mu| \geq \lambda\sigma) \leq \frac{1}{\lambda^2}$

- Another version $\quad Pr(X=0) \leq \frac{Var(X)}{E(X)^2}$

Pf $\quad Pr(X=0) \leq Pr(|X-\mu| \geq \mu) \leq \frac{\sigma^2}{\mu^2} = \frac{Var(X)}{E(X)^2}$

Corollary: If $Var(X) = o(E(X))^2$ then $Pr(X>0) = 1-o(1)$

Another 2nd moment inequality

$$\boxed{Pr(X>0) \geq \frac{(E(X))^2}{E(X^2)}}$$

Follows from Cauchy-Schwartz

$$\boxed{[E(XY)]^2 \leq E(X^2)\,E(Y^2)} \longrightarrow$$

Proof: wlog $E(X^2)>0$, $E(Y^2)>0$

Let $U = \frac{X}{\sqrt{E(X^2)}} \quad V = \frac{Y}{\sqrt{E(Y^2)}}$

$2|UV| \leq U^2 + V^2$

$\Rightarrow 2|E(UV)| \leq 2E(|UV|)$

$\leq E(U^2) + E(V^2) = 2$

$\Rightarrow [E(UV)]^2 \leq 1$

$\equiv [E(XY)]^2 \leq E(X^2)E(Y^2)$

Set $Y = 1_{X>0}$

$$[E(X)]^2 \leq E(X^2)\,\underbrace{E[(1_{X\geq 0})^2]}_{Pr(X>0)}$$

Lovász Local Lemma

Let $E_1, E_2, \ldots, E_n$ be set of "bad" events $\qquad Pr(E_i) < 1 \quad \forall i$

Say want to show $Pr\left(\bigcap_{i=1}^{m} \overline{E_i}\right) > 0$ "positive probability that nothing bad happens"

2 cases where easy:

1. E_i are mutually independent
2. $\sum_{i=1}^{n} Pr(E_i) < 1$ union bound suffices

LLL is clever comb

Defn E mutually indep of $E_1, \ldots, E_n$ if $\forall$ subset $I \subseteq [1..n]$

$$Pr\left(E \mid \bigcap_{j \in I} E_j\right) = Pr(E)$$

Defn A dependency graph for $E_1, E_2, \ldots, E_n$ is $G = (V, E)$ where $V = \{1, 2, \ldots, n\}$ & E_i is mutually indep of $\{E_j \mid (i,j) \notin E\}$

Lovász Local Lemma

Let $E_1, \ldots, E_n$ be set of events s.t.

1. $Pr(E_i) < p \quad \forall i$
2. The max degree in dependency graph is d
3. $4dp \leq 1$

Then $Pr\left(\bigcap_{i=1}^{n} \overline{E_i}\right) > 0$

several variants & generalizations (see notes)

Application: k-SAT

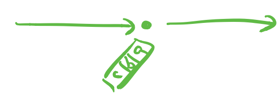
Let φ be a $\underbrace{\text{k-SAT formula}}_{\text{each clause has k literals}}$ w/ n vars
m clauses.

$\boxed{\text{Thm:}}$ Let φ be a k-SAT formula w/ n vars
m clauses.
If no var appears in $> T = \frac{2^k}{4k}$ clauses,
then formula has satisfying assignment

Application 2: Packet Routing

graph; n packets
each packet has
s_i source
t_i destination
and specific path P_i $\quad s_i \xrightarrow{P_i} t_i$

only one packet can traverse an edge per time unit

so 

Schedule specifies for each packet when to move, when to wait

$d = \max_i |P_i|$ dilation

$c = \max_e (\#\text{paths } P_i \text{ that use } e)$ congestion

How long for each packet to reach its destination?

$\Omega(c+d)$??? $O(cd)$

[Leighton, Rao, Maggs] $\exists$ schedule of length $O(c+d)$ always
indep of n!

Can be proved using LLL

High level idea:
for each packet, assign random initial delay
in $[1, \alpha(c+d)]$

guarantees limited dependency between congestion on different edges in different time periods

Algorithmic version

[Moser, Tardos]

2 clauses C_i & C_j are <u>dependent</u> if they share a var

$D(C_i) \triangleq \{C_j \mid C_i \text{ \& } C_j \text{ dependent}\}$ Let $d = \max_i |D(C_i)|$

Thm Let φ be a k-SAT formula with $d \le \frac{2^k}{8}$ (m clauses, n vars)
Then φ is satisfiable & a satisfying assignment can be found in poly time.

Super cool proof

Algorithm

Initialize $\vec{x} = (x_1, \dots, x_n)$
where $x_i = \begin{cases} T & \text{w.p. } \frac{1}{2} \\ F & \text{w.p. } \frac{1}{2} \end{cases}$

While $\exists$ clause C that is not satisfied
Fix(C)

Fix(C)

Randomly reassign k vars in C to T/F (indep w.prob $\frac{1}{2}$)
$\Rightarrow$ gives updated $\vec{x}$
While some clause D of φ that shares vars w/ C is violated
Fix(D)

[note D could be C]

Always process clauses in fixed order