2nd Moment Method

- Chebyshev's Inequality
  \[ \forall \lambda > 0 \quad \Pr( |X - \mu| > \lambda \sigma) \leq \frac{1}{\lambda^2} \]
- Another version
  \[ \Pr( X \geq 0 ) \leq \frac{ \text{Var}(X) }{ E(X) } \]
  \[ \Pr( X = 0 ) \leq \frac{ \text{Var}(X) }{ E(X) } \]

(Corollary: If \( \text{Var}(X) = o(E(X)^2) \) then \( \Pr(X > 0) = 1 - o(1) \))

Another 2nd moment inequality
\[ \Pr(X > 0) \geq \frac{(E(X))^2}{E(X^2)} \]

Follows from Cauchy-Schwartz
\[ [E(XY)]^2 \leq E(X^2) E(Y^2) \]

Proof: \( \text{wlog } \frac{E(X)}{E(Y)} > 0 \)

Let \( U = \frac{X}{\sqrt{E(X^2)}}, \quad V = \frac{Y}{\sqrt{E(Y^2)}} \)

\[ 2 |UV| \leq U^2 + V^2 \]
\[ \Rightarrow 2 |E(UV)| \leq 2 E(1UV) \]
\[ \leq E(U^2) + E(V^2) = 2 \]
\[ \Rightarrow [E(UV)]^2 \leq 1 \]
\[ \Rightarrow [E(XY)]^2 \leq E(X)^2 E(Y^2) \]
Lovász Local Lemma

Let $E_1, E_2, \ldots, E_n$ be a set of "bad" events $\Pr(E_i) < 1 \quad \forall i$

Say want to show $\Pr(\bigcap_{i=1}^n \overline{E_i}) > 0$ "positive probability that nothing bad happens"

2 cases where easy:
1. $E_i$ are mutually independent
2. $\sum_{i=1}^n \Pr(E_i) < 1$ union bound suffice

LLL is clever comb

Defn $E$ mutually indep across $E_1, E_2, \ldots, E_n$ if $Y$ subset $E \subseteq \{1, \ldots, n\}$

$$\Pr(E | \bigcap_{j \notin Y} E_j) = \Pr(E)$$

Defn A dependency graph for $E_1, E_2, \ldots, E_n$ is $G = (V, E)$

where $V = \{1, 2, \ldots, n\}$ & $E$ is mutually indep of $\{E_j | (i, j) \in E\}$

Lovász Local Lemma

Let $E_1, E_2, \ldots, E_n$ be a set of events s.t:
1. $\Pr(E_i) < 1 \quad \forall i$
2. The max degree in dependency graph is $d$
3. $4dp \leq 1$

Then $\Pr(\bigcap_{i=1}^n \overline{E_i}) > 0$

Several variants & generalizations (see notes)
Let \( \varphi \) be a \( k \)-SAT formula with \( n \) vars and \( m \) clauses.

Each clause has \( k \) literals.

Thm: Let \( \varphi \) be a \( k \)-SAT formula with \( n \) vars and \( m \) clauses.

If no variable appears in more than \( T = \frac{2^k}{4k} \) clauses, then the formula has a satisfying assignment.
Application 2: Packet Routing

- A graph with \( n \) packets
  - Each packet has a source \( s \) and destination \( t \)
- Only one packet can traverse an edge per time unit

Schedule specifies when to move, when to wait

\[
\begin{align*}
d &= \text{max}\left(\{P_i\} \right) \quad \text{dilation} \\
c &= \text{max}\left(\#\text{paths } P_i \text{ that use } e\right) \quad \text{congestion}
\end{align*}
\]

How long for each packet to reach its destination?

\[
\mathcal{L}(cd) \quad ???? \quad O(cd)
\]

[Leighton, Rao, Maggs] \( \exists \) schedule of length \( O(cd) \) always independent of \( n \)

Can be proved using LLL

High level idea:

- For each packet, assign random initial delay in \([1, \alpha(c+d)]\)

Guarantees limited dependency between congestion on different edges in different time periods
Algorithmic version

Algorithms

Algorithm

Initialize \( x^0 = (x_1, \ldots, x_n) \)
where \( x_i \in \{0, 1\} \)

While some clause \( C \) that is not satisfied

\[ \text{Fix}(C) \]

Randomly reassign \( k \) vars \( u \in C \)
    to \( +/\) (indep w/p \( \frac{1}{2} \))

\( \implies \) give updated \( x \)

While some clause \( D \) of \( \Psi \) that shares vars w/ \( C \) is violated

\[ \text{Fix}(D) \]

Fix \( D \) could be \( C \)

Thm

Let \( \Psi \) be a \( k \)-SAT formula with \( d \leq \frac{2^k}{3} \) (in \( \max \) clauses, vars)

Then \( \Psi \) is satisfiable & a satisfying assignment can be found in poly time.