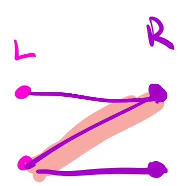
Bipartite Matching

Given bipartite graph (L, R, E)
a matching $M \subseteq E$ is a set of edges that share no common endpts

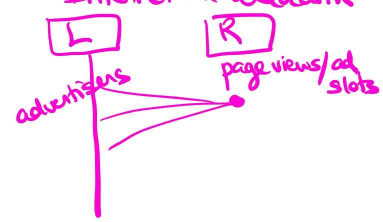
Goal: find matching of maximum size

Online bipartite matching

L is known ahead of time
vertices in R arrive one at a time
when $j \in R$ arrives
learn which vertices in L are neighbors of j
make an irrevocable decision as to which neighbor of any to match i to



Killer app:
Internet ad allocation



Greedy Alg

match arriving node to any available neighbor

Claim: Greedy always obtains a matching of size $\geq \frac{1}{2}$ OPT

Analysis: every time Greedy adds an edge to matching
think of it as earning \$1
Place 50¢ on each endpt

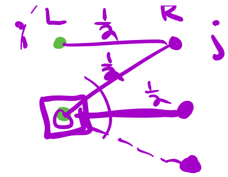
Now consider any edge (i, j) matched by OPT (when j arrives)
If Greedy can't match j, Then i is already matched.
"charge" OPT's edge to 50¢ on i

$\Rightarrow$ $\forall$ \$1 OPT earns, Greedy earns $\geq$ 50¢

the competitive ratio of Greedy $\left[:= \max_{I} \frac{\text{size of Greedy matching on } I}{\text{size of OPT matching on } I} \right]$
$= \frac{1}{2}$

And no deterministic alg can do better.
So we turn to randomization

Fractional matching problem



$\frac{3}{4}$ of OPT

When a node $j \in R$ arrives, can allocate it fractionally

x_{ij} fraction of j matched to i

constraints $\sum_{j \in R} x_{ij} \le 1$ total amt of i matched ≤ 1

$\sum_{i \in L} x_{ij} \le 1$ total amt of j matched ≤ 1

Claim: Let A be a randomized alg for integral matching

$\Rightarrow \exists$ deterministic fractional alg D s.t.

$\forall$ instance I

$$\sum_{i,j} x_{ij}^D(I) = E\Big(\sum_{i,j} \underbrace{X_{ij}^A(I)}\Big)$$

Indicator of event (i,j) matched by alg A on instance I

Proof: D "simulates A"

when j arrives, set $x_{ij}^D = \Pr(X_{ij}^A = 1)$

$\forall i \quad \sum_{j \in N(i)} X_{ij}^A \le 1 \quad \Rightarrow E\Big(\sum_{j \in N(i)} X_{ij}^A\Big) \le 1$

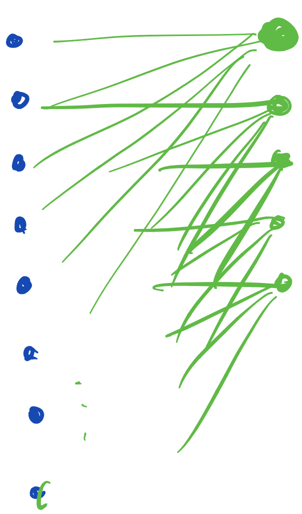
$\Rightarrow \sum_{j \in N(i)} x_{ij}^D \le 1$

$\Rightarrow \sum_{(i,j) \in E} x_{ij}^D = E\Big(\sum_{(i,j) \in E} X_{ij}^A\Big) = E(\text{performance of } A)$

$\Rightarrow$ upper bound on c.r. of deterministic fractional alg

$\Rightarrow$ upper bound on c.r. of randomized alg

det fractional



OPT max matching n.

i^{th} vertex on left.

$\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \dots \frac{1}{i}$

$= H_n - H_i = \ln\left(\frac{n}{i}\right)$

$i = \frac{n}{e}$ $\quad$ $\ln\left(\frac{n}{i}\right) = 1$

$n - i$

size of online mching $n\left(1 - \frac{1}{e}\right)$

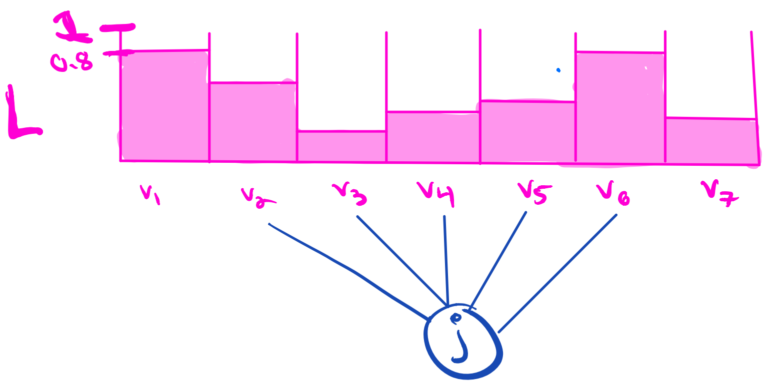
$\simeq 0.63\,n$

Water-level alg

Imagine vertices on LHS are water containers, with capacity 1

RHS - each vertex is source of 1 unit of water

When $j \in R$ arrives "fill up" neighbors in obvious way



pour 1 unit of water in

Primal dual analysis

$$\max \sum_{(i,j)\in E} x_{ij}$$

$$\sum_{j\in N(i)} x_{ij} \le 1 \qquad i\in L$$

$$\sum_{i\in N(j)} x_{ij} \le 1 \qquad j\in R$$

$$x_{ij} \geq 0$$

Summary of what we did: "online" primal-dual analysis

Primal

$$\max \sum_{(i,j)\in E} x_{ij}$$

$$\sum_{j\in N(i)} x_{ij} \le 1 \quad \forall i$$

$$\sum_{i\in N(j)} x_{ij} \le 1 \quad \forall j$$

$$x_{ij} \ge 0 \quad \forall (i,j)\in E$$

Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j$$

$$\alpha_i + \beta_j \ge 1 \quad \forall (i,j)\in E$$

$$\alpha_i \ge 0 \quad \forall i$$

$$\beta_j \ge 0 \quad \forall j$$

Weak duality $\Rightarrow$ feasible primal $\le \boxed{\text{OPT}} \le$ feasible dual

$$\sum_{(i,j)\in E} x_{ij} \le \boxed{\sum_{(i,j)\in E} x^*_{ij}} \le \sum_i \alpha_i + \sum_j \beta_j$$

As vertices arrived, we constructed near-feasible dual by ensuring that $\forall (i,j)\in E \quad \alpha_i + \beta_j \ge 1 - \frac{1}{e}$

$$\Rightarrow \quad \alpha_i' = \frac{\alpha_i}{1-\frac{1}{e}} \;\forall i \qquad \beta_j' = \frac{\beta_j}{1-\frac{1}{e}} \;\forall j$$

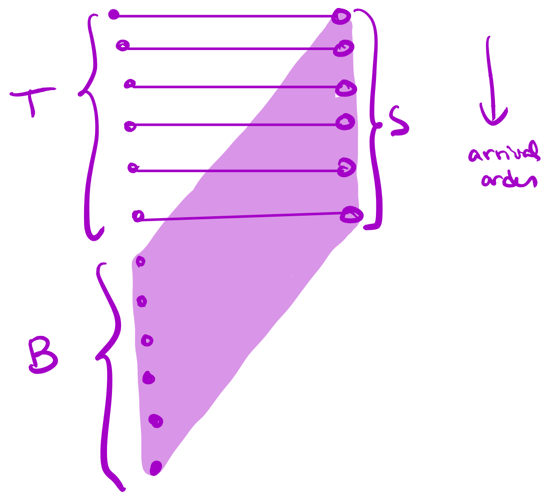
is feasible dual

$$\Rightarrow \text{OPT} \le \sum_i \hat{\alpha}_i + \sum_j \hat{\beta}_j = \frac{1}{\left(1-\frac{1}{e}\right)}\left[\sum_i \alpha_i + \sum \beta_j\right]$$

$$= \frac{1}{\left(1-\frac{1}{e}\right)} \sum_{(i,j)\in E} x_{ij}$$

$$\Rightarrow \quad \sum_{(i,j)\in E} x_{ij} \ge \left(1-\frac{1}{e}\right)\text{OPT}$$

Integral Online Matching



Ranking Algorithm • Select random total order π of elts of L
• when new vertex in R arrives, match it to highest ranked neighbor according to π,

[Karp, Vazirani, Vazirani] 1990

series of papers simplifying

[Devanur, Jain, Kleinberg] 2013

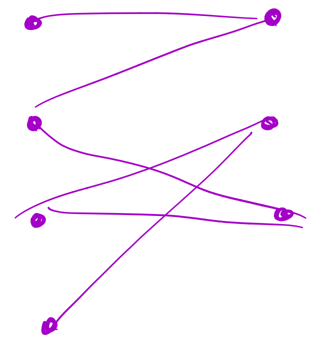
First idea: online fractional $1-\frac{1}{e}$ competitive.

every fractional matching is convex comb of integer soln s.

Can we maintain integral matching whose expected size = value of fractional matching produced by Water Level?

Fractional

Randomized



Linear programming duality

FYI

(P)

min $c_1x_1 + c_2x_2 + \dots + c_nx_n$

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \geq b_1$

$a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n \geq b_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \geq b_m$

$x_1, x_2, \dots x_n \geq 0$

(D)

max $b_1y_1 + b_2y_2 + \dots + b_my_m$

$a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m \leq c_1$

$a_{12}y_2 + a_{22}y_2 + \dots + a_{m2}y_m \leq c_2$

$\vdots$

$a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m \leq c_n$

$y_1, y_2, \dots, y_m \geq 0$

$$\boxed{\begin{array}{l} \min\ c \cdot x \\ \quad Ax \geq b \\ \quad x \geq 0 \end{array}}$$

$$\boxed{\begin{array}{l} \max\ b \cdot y \\ \quad y^TA \leq c^T \\ \quad y \geq 0 \end{array}}$$

$\vec{x}, \vec{c} \in \mathbb{R}^n$
$\vec{y}, \vec{b} \in \mathbb{R}^m$
$A \in \mathbb{R}^{m \times n}$

By construction OPT (D) $\leq$ OPT (P) "weak duality"

x feasible for (P), y feasible for (D)

$\Rightarrow b \cdot y \leq c \cdot x$ $\quad (y^Tb \leq y^TAx \leq c^Tx)$

$\Rightarrow$ if $by = cx$ both are optimal

__Duality Thm__ If (P) & (D) are primal-dual pair of LPs then one of following holds

1. both infeasible
2. (P) unbounded, (D) infeasible
3. (D) unbounded, (P) infeasible
4. both feasible & $\exists$ opt solns $\underbrace{x^*, y^* \text{ s.t. } cx^* = by^*}$

Complementary Slackness $\equiv$ $\forall i \quad x_i^*(c_i - \sum_j y_j^* a_{ji}) = 0$

$\forall j \quad y_j^*(\sum_i a_{ji} x_i^* - b_j) = 0$