Bipartite Matching

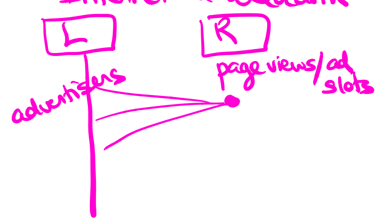
Given bipartite graph (L, R, E)
a matching $M \subseteq E$ is a set of edges that share no common endpts

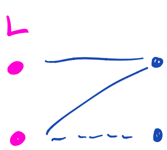
Goal: find matching of maximum size

Online bipartite matching

L is known ahead of time
vertices in R arrive one at a time
when $j \in R$ arrives
 learn which vertices in L are neighbors of j
 make an irrevocable decision as to
  which neighbor of any to match i to

Killer app:
Internet ad allocation





Greedy Alg

match arriving node to any available neighbor

Claim: Greedy always obtains a matching of size $\geq \frac{1}{2}$ OPT

Analysis: every time Greedy adds an edge to matching
 think of it as earning \$1
 Place 50¢ on each endpt

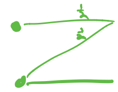
Now consider any edge (i,j) matched by OPT (when j arrives)
If Greedy can't match j, then i is already matched.
 "charge" OPT's edge to 50¢ on i

$\Rightarrow$ $\forall$ \$1 OPT earns, Greedy earns $\geq$ 50¢

The competitive ratio of Greedy $\left[:= \max_I \frac{\text{size of Greedy matching on } I}{\text{size of OPT matching on } I}\right]$
$= \frac{1}{2}$

And no deterministic alg can do better.
 So we turn to randomization

Fractional matching problem



When a node $j \in R$ arrives, can allocate it fractionally

x_{ij} fraction of j matched to i

constraints $\sum_{j \in R} x_{ij} \le 1$ total amt of i matched ≤ 1

$\sum_{i \in L} x_{ij} \le 1$ total amt of j matched ≤ 1

Claim: Let A be a randomized alg for integral matching

$\Rightarrow$ $\exists$ deterministic fractional alg D s.t.

$\forall$ instance I

$$\sum_{i,j} x_{ij}^D(I) = E\Big(\sum_{i,j} \underbrace{X_{ij}^A(I)}\Big)$$

Indicator of event (i,j) matched by alg A on instance I

Proof: D "simulates A"

when j arrives, set $x_{ij}^D = \Pr(X_{ij}^A = 1)$

$\forall i \quad \sum_{j \in N(i)} X_{ij}^A \le 1 \quad \Rightarrow E\Big(\sum_{j \in N(i)} X_{ij}^A\Big) \le 1$

$\Rightarrow \sum_{j \in N(i)} x_{ij}^D \le 1$

$$\Rightarrow \sum_{(i,j) \in E} x_{ij}^D = E\Big(\sum_{(i,j) \in E} X_{ij}^A\Big) = E(\text{performance of } A)$$

$\Rightarrow$ upper bound on c.r. of deterministic fractional alg

$\Rightarrow$ upper bound on c.r. of randomized alg

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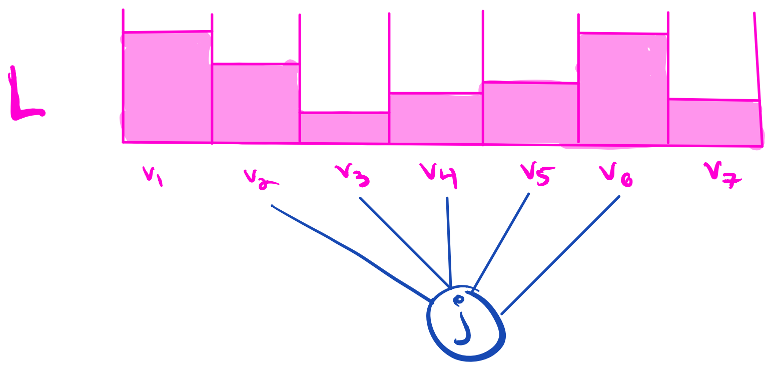
Water-level alg

Imagine vertices on LHS are water containers, with capacity 1

RHS - each vertex is source of 1 unit of water

When $j \in R$ arrives "fill up" neighbors in obvious way



pour 1 unit
of water in

Primal dual analysis

$$\max \sum_{(i,j)\in E} x_{ij}$$

$$\sum_{j\in N(i)} x_{ij} \leq 1 \qquad i\in L$$

$$\sum_{i\in N(j)} x_{ij} \leq 1 \qquad j\in R$$

$$x_{ij} \geq 0$$