__Today__

- finish SDP rounding
  3-coloring

- randomized online
  bipartite matching

$\boxed{\text{MAXCUT}}$

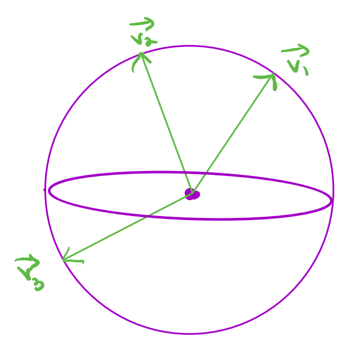
Input: $G=(V,E)$ $\qquad w_{ij} \quad \forall (i,j)\in E$

Goal: partition vertex set so as to max weight of endpts crossing cut.

$$\max \quad \frac{1}{2}\sum_{(i,j)\in E} w_{ij}\,(1-y_i y_j)$$
$$y_i^2 = 1 \qquad \forall i\in V$$

Vector programming relaxation

$$\max \quad \frac{1}{2}\sum_{(i,j)\in E} w_{ij}\,(1-\vec{v}_i\cdot\vec{v}_j)$$
$$\vec{v}_i\cdot\vec{v}_i = 1 \qquad \forall i\in V$$
$$\vec{v}_i \in \mathbb{R}^n$$



SDP rounding

Intro to semi-definite programming

linear programming where vars are entries in psd matrix

Defn If A is a symmetric n by n matrix
then A is a positive semidefinite (psd) matrix $\equiv A \succeq 0$
iff any of the following equivalent conditions hold

1. $\forall \vec{c} \in \mathbb{R}^n$, $\quad \vec{c}^T A \vec{c} \geq 0$
2. A has nonnegative eigenvalues
3. $A = V^T V$ for some $m \times n$ matrix V, $m \leq n$
4. $A = \sum_{i=1}^{n} \lambda_i x_i x_i^T$ for some $\lambda_i \geq 0$ and orthonormal vectors $x_i \in \mathbb{R}^n$

Semidefinite program (SDP)

max or min $\sum_{i,j} c_{ij} x_{ij}$

subject to $\sum_{i,j} a_{ijk} x_{ij} = b_k$

$x_{ij} = x_{ji} \quad \forall i,j$

$X = (x_{ij}) \succeq 0$

$\equiv$ Vector program

max or min $\sum_{i,j} c_{ij} (v_i \cdot v_j)$

subject to $\sum_{i,j} a_{ijk} (v_i \cdot v_j) = b_k$

$v_i \in \mathbb{R}^n \quad i = 1, \ldots, n$

given $X \Rightarrow X = V^T V$,
set v_i to be i^{th} col of V

Key fact:

SDPs can be solved to within additive error ε
in time poly(size of input, $\log(\frac{1}{\varepsilon})$)

in our discussions, we ignore additive error ε

MAXCUT

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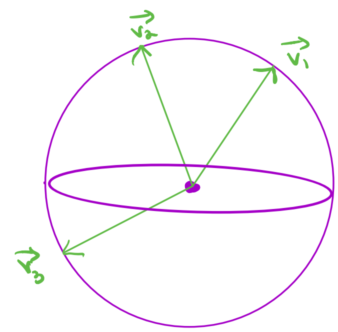


Vector programming relaxation

$$\max \quad \frac{1}{2}\sum_{(i,j)\in E} w_{ij}\left(1-\vec{v}_i\cdot\vec{v}_j\right)$$

$$\vec{v}_i\cdot\vec{v}_i = 1 \qquad \forall i \in V$$

$$\vec{v}_i \in \mathbb{R}^n$$

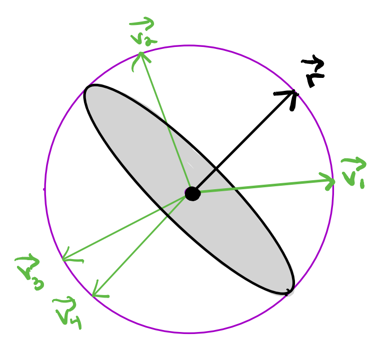
Can solve SDP in poly time.

$\boxed{\text{Claim:}}$ MAXCUT OPT $\leq$ SDP OPT

But how to round? get large contribution to OPT when $v_i\cdot v_j$ very -ve

Random hyperplane rounding

Solve SDP $\longrightarrow v_1^*, v_2^*, \ldots, v_n^*$
pick random hyperplane thru origin
partition vertices based on which side of hyperplane

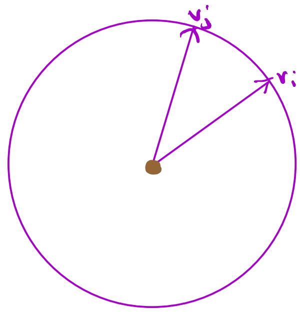


Put i into S if $v_i\cdot\vec{r} \geq 0$
i onto $\bar{S}$ if $v_i\cdot\vec{r} < 0$

$\Rightarrow \boxed{(z_1,\ldots,z_n)}$
$z_i \sim N(0,1)$

$$E(\text{weight of resulting cut } (S,\bar{S})) = \sum_{(i,j)\in E} w_{ij} \Pr((i,j) \text{ gets cut})$$

(i,j)

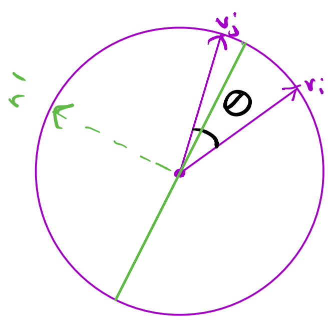


$$\vec{r} = \vec{r}\,' + \vec{r}\,''$$

$\vec{r}\,'$: projection of r to this plane

$\vec{r}\,''$: orthogonal.

$\frac{r'}{\|r'\|}$ is uniformly dist'd around circle



$\Pr((i,j) \text{ gets cut})$

$= \Pr(\text{unif random diam cuts between them})$

$$= \frac{\theta}{\pi} = \frac{\arccos(\vec{v}_i \cdot \vec{v}_j)}{\pi}$$

$$E(\text{weight of resulting cut } (S,\bar{S})) = \sum_{(i,j)\in E} w_{ij} \Pr((i,j) \text{ gets cut})$$

$$\forall x \;\; -1 \le x \le 1 \qquad \frac{\frac{\arccos(x)}{\pi}}{\frac{1-x}{2}} \geq \alpha$$

$$= \sum_{(i,j)\in E} w_{ij} \frac{\arccos(\vec{v}_i \cdot \vec{v}_j)}{\pi} \geq \alpha \underbrace{\sum_{(i,j)\in E} w_{ij} \frac{(1 - v_i \cdot v_j)}{2}}_{\text{obj of SDP}}$$

$$\alpha \geq \min_{-1 \le x \le 1} \frac{\frac{1}{\pi}\arccos(x)}{\frac{1}{2}(1-x)} \geq 0.878\ldots$$

$$E(\text{wt of cut produced by alg}) \geq 0.878\, OPT_{SDP} \geq 0.878\, OPT$$

Hardness

① If $\exists$ approx alg for MAXCUT with approx ratio ≥ 0.941, then $P = NP$.

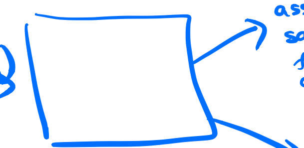
② If the "unique games conjecture" is true, there is no approx alg for MAXCUT with approx ratio better than $0.878\ldots$

Unique Games Conjecture

$\forall \varepsilon > 0$, $\exists$ prime q s.t.

$(1-\varepsilon, \varepsilon)$ Gap version

MAX2LIN(q) instance



if $\exists$ assignment satisfying $\geq 1-\varepsilon$ fraction of eqns output yes

if no assignment satisfies $> \varepsilon$ fraction output NO

admits no poly time soln unless $P = NP$

MAX2LIN(q)

q prime

input: linear equations mod q w/ unknowns $x_1, \ldots, x_n \in \{0, 1, \ldots, q-1\}$

(form $x_i - x_j = c$)

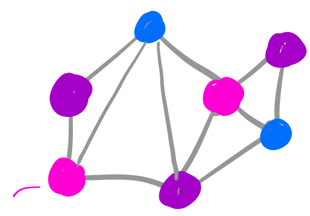
$$x_3 - x_{11} \equiv 87 \pmod{97}$$
$$x_7 - x_{22} \equiv 3 \pmod{97}$$
$$\vdots$$
$$x_7 - x_{19} \equiv 56 \pmod{97}$$

Problem: Find assignment of x_i's that satisfies max possible # of eqns

③ Int gap of the [GW] SDP $= 0.878\ldots$

③ Every poly sized LP relaxation of MAXCUT has integrality gap of $\frac{1}{2}$.

3-Coloring a 3-colorable graph



Given graph $G=(V,E)$

& <u>promise</u> that it is 3-colorable

What is min k s.t. we can find a k-coloring of G in poly time?

<u>Simple results:</u>

① A graph with max degree Δ can be colored with $\leq \Delta+1$ colors

② A 3-colorable graph can be colored with $O(\sqrt{n})$ colors.

Find a vertex of deg $\geq \sqrt{n}$
Use 3 colors to color it & its neighbors (neighborhood 2-colorable)
Remove it & its neighbors from graph

An SDP-based alg

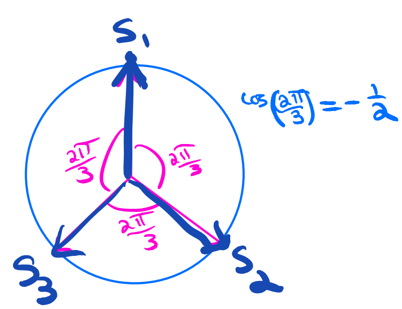
$$\begin{aligned}
\min \quad & \lambda \\
\text{s.t.} \quad & \vec{v}_i \cdot \vec{v}_j \leq \lambda & \forall (i,j) \in E \\
& \vec{v}_i \cdot \vec{v}_i = 1 & \forall i \in V \\
& \vec{v}_i \in \mathbb{R}^n
\end{aligned}$$

<u>Claim:</u>

if graph is 3-colorable

$\lambda \leq -\frac{1}{2}$

$\Rightarrow$



$$\min \lambda$$

$$\text{s.t.} \quad \vec{v}_i \cdot \vec{v}_j \le \lambda \quad \forall (i,j) \in E$$

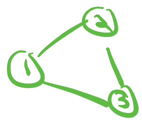
$$\vec{v}_i \cdot \vec{v}_i = 1 \quad \forall i \qquad (*)$$

$$\vec{v}_i \in \mathbb{R}^n \quad \forall i$$

Claim: if graph is 3-colorable $\lambda \le -\frac{1}{2}$

Aside: If G has a triangle, then optimal soln to SDP has $\lambda^* \ge -\frac{1}{2}$

Proof: Suppose



$$0 \le \left(\vec{v}_1+\vec{v}_2+\vec{v}_3,\ \vec{v}_1+\vec{v}_2+\vec{v}_3\right) = \vec{v}_1\cdot\vec{v}_1 + \vec{v}_2\cdot\vec{v}_2 + \vec{v}_3\cdot\vec{v}_3$$

$$+ \vec{v}_1\cdot\vec{v}_2 + \vec{v}_1\cdot\vec{v}_3 + \vec{v}_2\cdot\vec{v}_1 + \vec{v}_2\cdot\vec{v}_3 + \vec{v}_3\cdot\vec{v}_1 + \vec{v}_3\cdot\vec{v}_2$$

$\forall G$

$$\min \lambda$$

$$v_i \cdot v_j \le \lambda \quad \forall (i,j) \in E$$

$$v_i \cdot v_i = 1 \quad \forall i$$

$$v_i \in \mathbb{R}^n$$

If G k-colorable $\lambda^* \le -\frac{1}{k-1}$

If G has k-clique $\lambda^* \ge -\frac{1}{k-1}$

Defn $\theta(G) = 1 - \frac{1}{\lambda^*}$

called Lovasz Theta fn
(= k if G is k-colorable & has k-clique)

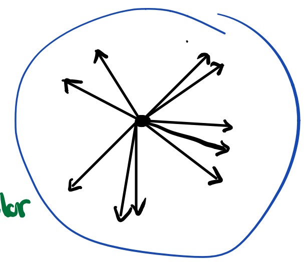
clique # of G $\le \theta(G) \le$ chromatic # of G

Back to coloring 3-colorable graphs

$$(*) \quad \begin{aligned} &\min \ \lambda \\ &v_i \cdot v_j \le \lambda \qquad \forall (i,j) \in G \\ &v_i \cdot v_i = 1 \qquad \forall i \\ &v_i \in \mathbb{R}^n \end{aligned}$$

<u>Algorithm</u>

1. Solve SDP $(*)$ $\Rightarrow$ v_i^* $i=1,\dots,n$
2. Choose t random hyperplanes thru origin
3. Color vertices in each region w/ diff color
4. remove any edges properly colored
5. Repeat steps 2–4 until have proper coloring



One execution of step 2 uses 2^t colors.

Goal: produce $\underbrace{\text{semi-coloring}}$ w.p. $\ge \frac{1}{2}$ $\qquad (**)$

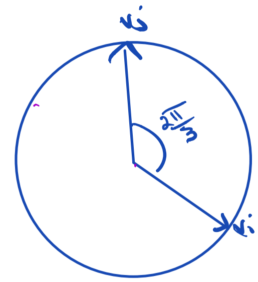
coloring of nodes s.t.
$\le \frac{n}{4}$ edges have same color at both endpts
$\Rightarrow$ at least $\frac{n}{2}$ vertices properly colored.

Observation: if k-colors sufficient to get semi-coloring,
$\Rightarrow$ graph can be properly colored with $O(k \log n)$ colors

What should t be to guarantee $(**)$?

Fix $(i,j) \in E$

$Pr(i \,\&\, j$ get same color$)$



$\Rightarrow E(\#$ edges with same color$)$

Let Δ^* be a parameter

1. Pick a vertex of deg $\geq \Delta^*$ & 3-color it & neighbors
2. Repeat step 1 until all vertices have degree $\leq \Delta^*$
3. Run SDP-based alg to color rest

Current best: $O(n^{0.199})$

NP-hard to color with 4 colors

$\boxed{\text{Huge open problem:}}$ Is there an alg for 3-coloring a 3-colorable graph that uses polylog n colors?