Today

- finish SDP rounding 3-coloring
- randomized online bipartite matching

MAXCUT

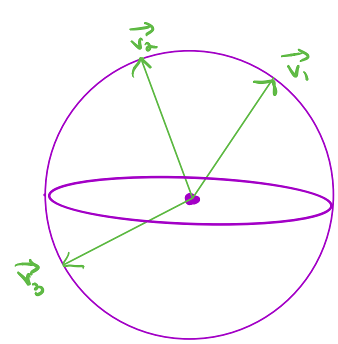
Input: $G=(V,E)$ $\quad w_{ij} \quad \forall (i,j)\in E$

Goal: partition vertex set so as to max weight of endpts crossing cut.

$$\max \quad \frac{1}{2}\sum_{(i,j)\in E} w_{ij}\,(1-y_i y_j)$$
$$y_i^2 = 1 \qquad \forall i\in V$$

Vector programming relaxation

$$\max \quad \frac{1}{2}\sum_{(i,j)\in E} w_{ij}\,(1-\vec{v}_i\cdot\vec{v}_j)$$
$$\vec{v}_i\cdot\vec{v}_i = 1 \qquad \forall i\in V$$
$$\vec{v}_i \in \mathbb{R}^n$$



SDP rounding

Intro to semi-definite programming

linear programming where vars are entries in psd matrix

Defn If A is a symmetric n by n matrix
then A is a positive semidefinite (psd) matrix $\equiv A \succeq 0$
iff any of the following equivalent conditions hold

1. $\forall \vec{c} \in \mathbb{R}^n$, $\vec{c}^T A \vec{c} \geq 0$
2. A has nonnegative eigenvalues
3. $A = V^T V$ for some $m \times n$ matrix V, $m \leq n$
4. $A = \sum_{i=1}^{n} \lambda_i x_i x_i^T$ for some $\lambda_i \geq 0$ and orthonormal vectors $x_i \in \mathbb{R}^n$

Semidefinite program (SDP)

max or min $\sum_{i,j} c_{ij} x_{ij}$

subject to $\sum_{i,j} a_{ijk} x_{ij} = b_k$

$x_{ij} = x_{ji} \quad \forall i,j$

$X = (x_{ij}) \succeq 0$

$\equiv$ Vector program

max or min $\sum_{i,j} c_{ij} (v_i \cdot v_j)$

subject to $\sum_{i,j} a_{ijk} (v_i \cdot v_j) = b_k$

$v_i \in \mathbb{R}^n \quad i = 1, \ldots, n$

given $X \Rightarrow X = V^T V$; set v_i to be i^{th} col of V

Key fact:

SDPs can be solved to within additive error ε in time poly(size of input, $\log(\frac{1}{\varepsilon})$)

in our discussions, we ignore additive error ε

$\boxed{\text{MAXCUT}}$

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Vector programming relaxation

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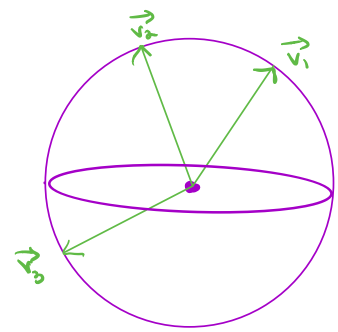
$$\vec{v}_i\in\mathbb{R}^n$$



Can solve SDP in poly time.

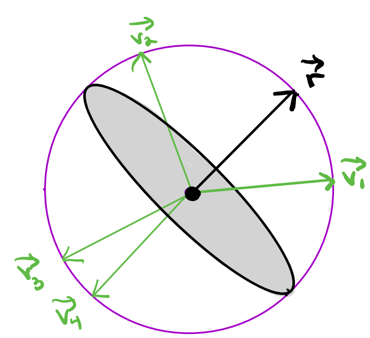
$\boxed{\text{Claim:}}$ MAXCUT OPT $\leq$ SDP OPT

But how to round? get large contribution to OPT when $v_i\cdot v_j$ very -ve

$\boxed{\text{Random hyperplane rounding}}$

Solve SDP $\longrightarrow$ $v_1^*, v_2^*, \ldots, v_n^*$
pick random hyperplane thru origin
partition vertices based on which side of hyperplane

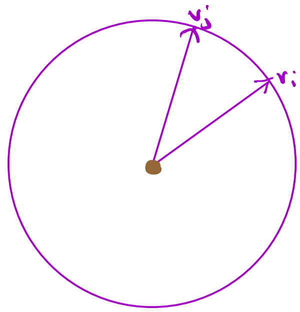
Put i into S if $v_i\cdot\vec{r}\geq 0$
i onto $\bar{S}$ if $v_i\cdot\vec{r}<0$



$\Rightarrow \boxed{(z_1,\ldots,z_n)}$
$z_i\sim N(0,1)$

$$E(\text{weight of resulting cut } (S,\bar{S})) = \sum_{(i,j)\in E} w_{ij} \Pr((i,j) \text{ gets cut})$$

(i,j)

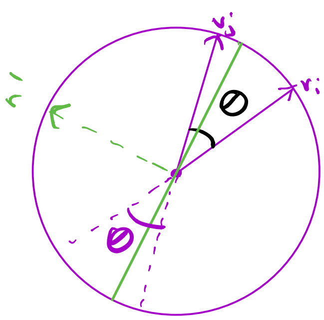


$\vec{r} = \vec{r}' + \vec{r}''$

$\vec{r}'$: projectn of r to this plane

$\vec{r}''$: orthogonal.

$\dfrac{r'}{\|r'\|}$ is uniformly dist'd around circle



$\Pr((i,j) \text{ gets cut})$

$= \Pr(\text{unif random diam cuts between them})$

$$\frac{2\theta}{2\pi} = \frac{\theta}{\pi} = \frac{\arccos(\vec{v}_i \cdot \vec{v}_j)}{\pi}$$

$$E(\text{weight of resulting cut } (S,\bar{S})) = \sum_{(i,j)\in E} w_{ij} \Pr((i,j) \text{ gets cut})$$

$$= \sum_{(i,j)\in E} w_{ij} \frac{\arccos(\vec{v}_i \cdot \vec{v}_j)}{\pi} \geq \alpha \underbrace{\sum_{(i,j)\in E} w_{ij} \frac{(1 - v_i \cdot v_j)}{2}}_{\text{obj of SDP}}$$

$\forall x \quad -1 \le x \le 1$

$$\frac{\frac{\arccos(x)}{\pi}}{\frac{1-x}{2}} \geq \alpha$$

$$\alpha = \min_{-1 \le x \le 1} \frac{\frac{1}{\pi}\arccos(x)}{\frac{1}{2}(1-x)} = 0.878\ldots$$

$$E(\text{wt of cut produced by alg}) \geq 0.878 \, OPT_{SDP} \geq 0.878 \, OPT$$

Hardness

① If $\exists$ approx alg for MAXCUT with approx ratio ≥ 0.941, then $P = NP$.

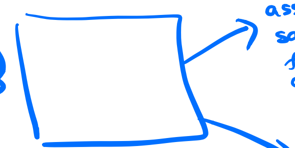
② If the "unique games conjecture" is true, there is no approx alg for MAXCUT with approx ratio better than $0.878\ldots$

Unique Games Conjecture

$\forall \varepsilon > 0$, $\exists$ prime q s.t.

$(1-\varepsilon, \varepsilon)$ Gap version

MAX2LIN(q) instance



if $\exists$ assignment satisfying $\geq 1-\varepsilon$ fraction of eqns output Yes

if no assignment satisfies $> \varepsilon$ fraction output NO

admits no poly time soln unless $P = NP$

MAX2LIN(q)

q prime

input: linear equations mod q w/ unknowns $x_1, \ldots, x_n \in \{0, 1, \ldots, q-1\}$

(form $x_i - x_j = c$)

$x_3 - x_{11} \equiv 87 \pmod{97}$

$x_7 - x_{22} \equiv 3 \pmod{97}$

$\vdots$

$x_7 - x_{19} \equiv 56 \pmod{97}$

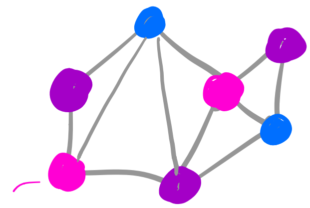
Problem: Find assignment of x_i's that satisfies max possible # of eqns

③ Int gap of the [GW] SDP $= 0.878\ldots$

③ Every poly sized LP relaxation of MAXCUT has integrality gap of $\frac{1}{2}$.

3-Coloring a 3-colorable graph

$\Rightarrow$ polytime alg guarantees k colors $\forall$ G s.t. G is 3-color



Given graph $G=(V,E)$

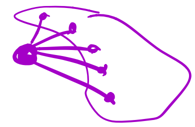
& promise that it is 3-colorable

What is min k s.t. we can find a k-coloring of G in poly time?

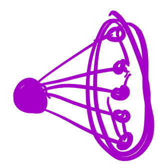
Simple results:

① A graph with max degree Δ can be colored with $\leq \Delta+1$ colors

Greedy



② A 3-colorable graph can be colored with $O(\sqrt{n})$ colors.



Find a vertex of deg $\geq \sqrt{n}$

Use 3 colors to color it & its neighbors (neighborhood 2-colorable)

Remove it & its neighbors from graph

$\leq \sqrt{n}$ times

$3\sqrt{n}$ colors

by ① remaining graph can be colored w/ $O(\sqrt{n})$ colors.

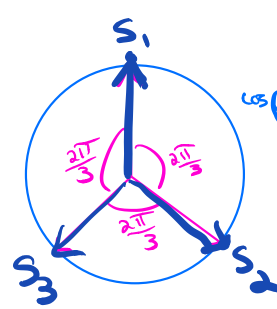
An SDP-based alg

$$\begin{aligned} \min\ & \lambda \\ \text{s.t.}\ & \vec{v}_i \cdot \vec{v}_j \leq \lambda && \forall (i,j) \in E \\ & \vec{v}_i \cdot \vec{v}_i = 1 && \forall i \in V \\ & \vec{v}_i \in \mathbb{R}^n \end{aligned}$$

Claim: if graph is 3-colorable $\lambda \leq -\frac{1}{2}$

v_i vector corresponding to node i

$$\lambda = \max_{\substack{i,j \\ \text{s.t. } (i,j) \in E}} \cos(\text{angle bet } v_i \,\&\, v_j) \Longrightarrow$$



$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

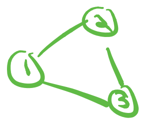
$$\min \ \lambda$$
$$\text{s.t.} \quad \vec{v}_i \cdot \vec{v}_j \le \lambda \quad \forall (i,j) \in E$$
$$\vec{v}_i \cdot \vec{v}_i = 1 \quad \forall i \qquad (*)$$
$$\vec{v}_i \in \mathbb{R}^n \quad \forall i$$

Claim: if graph is 3-colorable $\lambda \le -\frac{1}{2}$

Aside

If G has a triangle, then optimal soln to SDP has $\lambda^* \ge -\frac{1}{2}$

Proof: Suppose



$$0 \le (\vec{v}_1+\vec{v}_2+\vec{v}_3, \vec{v}_1+\vec{v}_2+\vec{v}_3) = \underbrace{\vec{v}_1\cdot\vec{v}_1 + \vec{v}_2\cdot\vec{v}_2 + \vec{v}_3\cdot\vec{v}_3}_{=3}$$
$$\underbrace{+ \vec{v}_1\cdot\vec{v}_2 + \vec{v}_1\cdot\vec{v}_3 + \vec{v}_2\cdot\vec{v}_1 + \vec{v}_2\cdot\vec{v}_3 + \vec{v}_3\cdot\vec{v}_1 + \vec{v}_3\cdot\vec{v}_2}_{\ge -3}$$

$$\boxed{\begin{array}{ll} \forall G & \\ \min \ \lambda & \\ v_i \cdot v_j \le \lambda & \forall (i,j) \in E \\ v_i \cdot v_i = 1 & \forall i \\ v_i \in \mathbb{R}^n & \end{array}}$$

Claim:
If G k-colorable $\lambda^* \le -\frac{1}{k-1}$
If G has k-clique $\lambda^* \ge -\frac{1}{k-1}$

Defn $\theta(G) = 1 - \frac{1}{\lambda^*}$

called Lovasz Theta fn
($=k$ if G is k-colorable & has k-clique)

$\lambda^*(G) = -\frac{1}{k-1}$
$\Rightarrow \theta(G) = k$

$$\text{clique \# of } G \le \theta(G) \le \text{chromatic \# of } G$$

Back to coloring 3-colorable graphs

$G \quad V = \{1, 2, \ldots, n\}$

$$(*) \quad \boxed{\begin{array}{ll} \min & \lambda \\ & v_i \cdot v_j \le \lambda \quad \forall (i,j) \in E \\ & v_i \cdot v_i = 1 \quad \forall i \\ & v_i \in \mathbb{R}^n \end{array}}$$

n vectors

$v_1^*, \ldots, v_n^*$

3-colorable

$\Rightarrow \lambda^* \le -\frac{1}{2}$

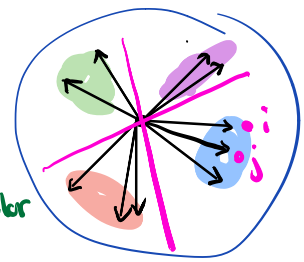
$\equiv$ if $(i,j) \in E$

$\theta \ge \frac{2\pi}{3}$

angle between v_i & v_j

Algorithm

1. Solve SDP $(*)$ $\Rightarrow$ v_i^* $i=1,\ldots,n$
2. Choose t random hyperplanes thru origin
3. Color vertices in each region w/ diff color
4. remove any edges properly colored
5. Repeat steps 2-4 with new colors until have proper coloring



One execution of step 2 uses 2^t colors.

Goal: produce $\underbrace{\text{semi-coloring}}$ w.p. $\ge \frac{1}{2}$ $(**)$

coloring of nodes s.t.
$\le \frac{n}{4}$ edges have same color at both endpts

$\Rightarrow$ at least $\frac{n}{2}$ vertices properly colored.

Observation: if k-colors sufficient to get semi-coloring,

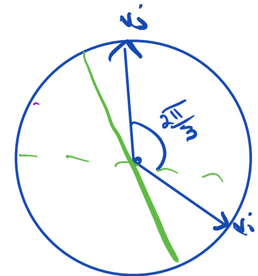
$\Rightarrow$ graph can be properly colored with $O(k \log n)$ colors

What should t be to guarantee $(**)$?

Fix $(i,j) \in E$

$Pr(i \,\&\, j \text{ get same color}) \le \left(1 - \frac{1}{\pi}\frac{2\pi}{3}\right)^t$

$= \frac{1}{3^t}$



$\frac{2\pi}{3\pi}$

Let Δ be max deg in graph.

$\Rightarrow E(\#\text{ edges with same color}) \le \frac{|E|}{3^t} \le \frac{n\Delta}{2\, 3^t} \overset{?}{=} \frac{n}{8}$

$3^t = 4\Delta \equiv \underbrace{t = \log_3(4\Delta)}$

By Markov Ineq

$Pr\left(\#\text{edges w/ same color} \ge \frac{n}{4}\right) \le \frac{\frac{n}{8}}{\frac{n}{4}} \le \frac{1}{2}.$

use $\quad O\left(\underline{\log n}\; 2^{\log_3(4\Delta)}\right) = \tilde{O}\left(\Delta^{\log_3 2}\right)$

$\Delta = \Omega(n)$

$\Downarrow$

$\tilde{O}\left(n^{\log_3 2}\right)$

$\Longrightarrow \; = \tilde{O}\left(n^{0.631}\right)$

#colors.

Let Δ^* be a parameter

1. Pick a vertex of deg $\geq \Delta^*$ & 3-color it & neighbors
2. Repeat step 1 until all vertices have degree $\leq \Delta^*$

$\Big\} \; 3 \cdot \frac{n}{\Delta^*}$

3. Run SDP-based alg to color rest

graph w/ max deg Δ^*

$\tilde{O}\left(\Delta^{*\log_3 2}\right)$ colors.

choose Δ^* to min $\frac{3n}{\Delta^*} + \Delta^{*\log_3 2}$

minimized when $\Delta^* = n^{\log_6 3}$

$\Rightarrow \tilde{O}\left(n^{0.39}\right)$ colors.

Current best: $O\left(n^{0.199}\right)$

NP-hard to color with 4 colors

$\boxed{\text{Huge open problem:}}$ Is there an alg for 3-coloring a 3-colorable graph that uses polylog n colors?