Optimization

min/max $f(x_1, \dots, x_n)$
subject to constraints

When can we solve efficiently?

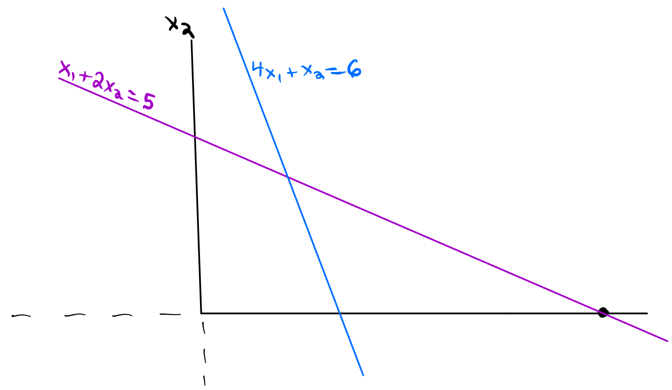
Today

- randomized rounding of LPs
  - MAX 2SAT
  - congestion min
- project info

Linear programming

L

$f(x_1, \dots, x_n)$ linear fn.
constraints linear



Example: Diet Problem

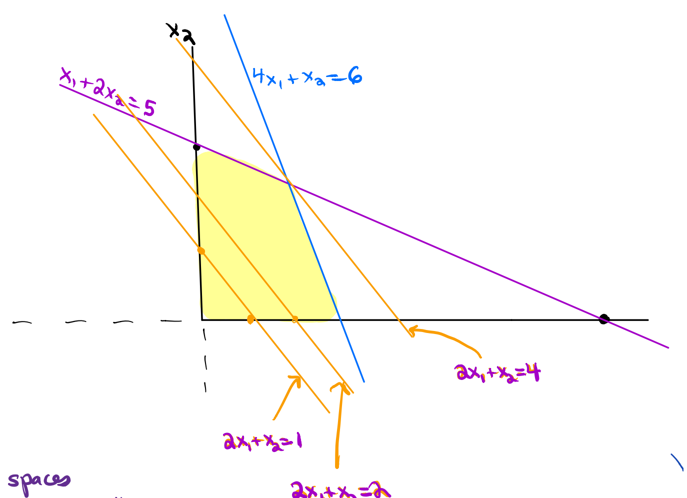
Athlete wants to max protein consumption
subject to ≤ 5 units of fat/day
$\leq \$6$ / day

	protein/lb	fat/lb	$/lb
steak	2	1	4
peanut butter (PB)	1	2	1

x_1 # lbs of steak/day
x_2 # lbs of PB/day

max $\boxed{2x_1 + x_2}$ objective fn
subject to
$$\boxed{\begin{array}{l} 4x_1 + x_2 \leq 6 \\ x_1 + 2x_2 \leq 5 \\ x_1, x_2 \geq 0 \end{array}}$$
feasible set
feasible region

feasible pt with max objective fn value is "optimal solution"



feasible set is "polyhedron": intersection of half spaces
if also bounded & nonempty ⇒ "polytope"

feasible set convex

linear cost fns define family of parallel hyperplanes
optimal feasible pt must occur at corner, a.k.a. vertex
[can't be expressed as convex comb of feasible pts]

Unfortunately too many vertices to enumerate $\approx m^{n/2}$
m constraints, n variables

Ex: $0 \leq x_i \leq 1$ $1 \leq i \leq n$
n dimensional hypercube

Input to LP problem

$$\boxed{\begin{array}{l}\vec{c} \in \mathbb{R}^n \\ A \in \mathbb{R}^{m\times n} \\ \vec{b} \in \mathbb{R}^m\end{array}}$$

find $x \in \mathbb{R}^n$ to

max $c \cdot x$

subject to $Ax \leq b$

$x_i \geq 0 \;\; \forall i$

$\equiv$ max $c_1x_1 + c_2x_2 + \dots + c_nx_n$

s.t. $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$

$x_1, x_2, \dots, x_n \geq 0$

max $\boxed{2x_1 + x_2}$

subject to $\boxed{\begin{array}{l}4x_1 + x_2 \leq 6 \\ x_1 + 2x_2 \leq 5 \\ x_1, x_2 \geq 0\end{array}}$

$c = \begin{pmatrix}2\\1\end{pmatrix}$

$A = \begin{pmatrix}4 & 1\\ 1 & 2\end{pmatrix}$

$b = \begin{pmatrix}6\\5\end{pmatrix}$

$a \cdot x \geq b \;\; \equiv \;\; -a \cdot x \leq -b$

$a \cdot x = b \;\; \equiv \;\; \begin{array}{l} a\cdot x \geq b \\ a \cdot x \leq b\end{array}$

x unconstrained $\quad x = x^+ - x^-$

$x^+, x^- \geq 0$

There are efficient algs for LP.

- simplex $\left.\right\}$ exp worst case, "smoothed" poly time
- ellipsoid
- interior pt methods

$\left.\right\}$ polynomial time (ellipsoid, interior pt methods)

$\boxed{\text{ellipsoid}}$: sometimes even exponential-sized LPs can be solved in poly time

poly time separation oracle

$\vec{z} \in \mathbb{R}^n \rightarrow \boxed{\text{sep oracle}}$ $\rightarrow$ yes $\vec{z}$ feasible

$\rightarrow$ no + sep hyperplane $a \cdot x \leq b$

every feasible x satisfies $a \cdot x \leq b$

but z doesn't $a \cdot z > b$

$\boxed{\text{Bottom line:}}$ If you can formulate your problem as LP, it can be solved in time $\text{poly}(m, n, \log(\max_{i,j}(a_{ij}, b_i, c_j)))$

$$\boxed{\text{Using "linear-programming relaxations" to get approx algs for NP-hard problems \& then apply randomized rounding}}$$

MAXSAT

Given a Boolean formula in CNF (and of ors)

n Boolean vars $\quad x_1, \dots, x_n$

m clauses $\quad C_1, \dots, C_m$

each clause is OR of some # of $\underbrace{\text{vars \& negations}}_{\text{literals}}$ $\quad$ e.g. $x_3 \vee \bar{x}_5 \vee x_{11}$

max-version even when all clauses have ≤ 2 literals

w_j weight of j^{th} clause

MAXSAT: find assignment that maximizes wt of satisfied clauses

MAX k SAT: all clauses have $\leq k$ literals

MAX EkSAT: all clauses have exactly k literals

Random assignment

assign $x_i \to T$ wp $\frac{1}{2}$
$\quad\quad\quad\;\, \to F$ wp $\frac{1}{2}$

k	Pr(clause sat)
1	$\frac{1}{2}$
2	3/4
3	7/8
4	15/16

Pr(clause with k literals satisfied) $= 1 - \frac{1}{2^k}$

MAX 2SAT $\quad E(\#\text{clauses satisfied})$
$$= \frac{N_1}{2} + N_2 \cdot \frac{3}{4}$$

MAX E3SAT $\quad E(\#\text{clauses satisfied}) = \frac{7}{8}$ total

No better approx is possible unless $P = NP$.

We will consider MAX 2-SAT $\quad$ all clauses have 1 or 2 literals

$\boxed{\text{Step 1:}}$ Formulate MAX-2SAT as integer linear program where we require all vars to be $\in \{0,1\}$

Suppose input formula has n vars, m clauses

Define vars $\quad x_1, \dots, x_n \quad \forall$ var in formula
$\quad\quad\quad\quad\quad\; z_1, \dots, z_m \quad \forall$ clause in formula

where $\quad x_i = \begin{cases} 1 & \text{var } i \; T \\ 0 & \text{o.w.} \end{cases} \quad z_j = \begin{cases} 1 & \text{if clause } j \text{ satisfied} \\ 0 & \text{o.w.} \end{cases}$

Ex
$C_j = (\bar{x}_1 \vee x_2)$
$\boxed{(1-x_1) + x_2 \geq z_j}$
$C_j = (x_5)$
$x_5 \geq z_j$

$$\max \sum_{j=1}^{m} w_j z_j$$

w_j weight of clause j ≥ 0

(1) $\quad \sum_{i \in C_j} y_{ij} \geq z_j \quad \forall$ clause j

$y_{ij} = \begin{cases} x_i & \text{if var occurs +vely in } C_j \\ 1 - x_i & \text{-vely} \end{cases}$

$$\boxed{\begin{array}{l} x_i \in \{0,1\} \\ z_j \in \{0,1\} \end{array}}$$

ILP is NP-complete

integer linear programming is NP-hard though!

Step 2: Relax to a LP and solve $\Rightarrow$ $\underbrace{z_1^*, \dots z_m^*, x_1^*, \dots, x_n^*}_{\text{OPT of (2)}}$ fractional

$\sum_{j=1}^{m} w_j z_j^*$

$$\max \sum_{j=1}^{m} w_j z_j$$

(2)

$$\sum_{i \in C_j} y_{ij} \geq z_j \qquad j \in 1, \dots, m$$

where $y_{ij} = \begin{cases} x_i & \text{if var } i \text{ pos in } C_j \\ 1 - x_i & \text{" neg in } C_j \end{cases}$

$x_i \in \{0,1\}$ $i = 1..n$

$z_j \in \{0,1\}$ $j = 1..m$

$\Rightarrow$ $0 \leq x_i \leq 1$ $\forall i$

$0 \leq z_j \leq 1$ $\forall j$

Can solve in poly time

Key observation: $\underset{(2)}{\underline{OPT_{LPR}}} \geq \underset{(2)}{OPT_{ILP}}$

$\Downarrow$

$\boxed{\sum_{j=1}^{m} w_j z_j^*}$

Step 3: Round the soln $\Rightarrow$ integer soln

Cool idea: treat x_j^* as probabilities

Set var i $= \begin{cases} T & \text{w.prob } x_i^* \\ F & \text{o.w.} \end{cases}$ $\Big\}$ indep for each var.

$\Uparrow$

Claim: $\forall j$ $\Pr(\text{clause } j \text{ satisfied}) \geq \frac{3}{4} z_j^*$

$$E\left(\sum_{j=1}^{m} w_j \mathbb{1}_{C_j \text{ satisfied by rounded soln}}\right) \geq \sum_{j=1}^{m} w_j \frac{3}{4} z_j^* = \frac{3}{4} \underline{OPT_{LPR}}$$

$$\geq \frac{3}{4} OPT^2$$

Proof of claim

suppose constraint for C_j is $\boxed{\begin{array}{c} x_r + x_s \geq z_j \\ 0 \leq z_j \leq 1 \end{array}}$ $(x_r \vee x_s)$

$\Rightarrow z_j^* = \min(1, \underline{x_r^* + x_s^*})$

$$\Pr(C_j \text{ is satisfied}) = 1 - (1-x_r^*)(1-x_s^*)$$
$$= x_r^* + x_s^* - x_r^* x_s^*$$
$$\geq x_r^* + x_s^* - \frac{1}{4}(x_r^* + x_s^*)^2$$
$$= z_j^*\left(1 - \frac{z_j^*}{4}\right) \quad \text{(using } x_r^* + x_s^* \le 1\text{)}$$
$$\underbrace{\left(1-\frac{z_j^*}{4}\right)}_{\geq \frac{3}{4}}$$
$$\geq \frac{3}{4} z_j^*$$

if $\underline{x_r^* + x_s^* \geq 1}$ (a)

$a - \frac{a^2}{4}$

$a\left(1 - \frac{a}{4}\right) \geq \frac{3}{4}$

$\geq \frac{3}{4} z_j^*$

$a_1, a_2 \geq 0$

$\frac{a_1+a_2}{2} \geq \sqrt{a_1 a_2}$

$\left(\frac{a_1+a_2}{2}\right)^2 \geq a_1 a_2$

$-a_1 a_2 \geq -\left(\frac{a_1+a_2}{2}\right)^2$

- Similar analysis shows randomized rounding gives $1-\frac{1}{e}$ approx alg for MAXSAT 0.63

- Choosing better of
  - randomized rounding soln
  - unbiased randomized alg

  $\Rightarrow \frac{3}{4}$ approx alg

- Solving LP & then setting x_i: True w.p. $f(x_i^*)$, False o.w., where $f: [0,1] \to [0,1]$, $1 - 4^{-x} \leq f(x) \leq 4^{x-1}$

  gives $\frac{3}{4}$ approx alg.

Best possible if we compare our solution against OPT_{LP}

Int gap of LP relaxation $\boxed{\max_{\text{instances}}}$ $\frac{\text{OPT integer soln}}{\text{OPT fractional soln.}}$

For MAX2SAT int gap is $\frac{3}{4}$

Ex: $(x_1 \vee x_2) \wedge (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2)$

$x_1 + x_2 \geq z_1$ $x_1 + (1-x_2) \geq z_2$ $(1-x_1) + (1-x_2) \geq z_4$

OPT int can satisfy 3 clauses.

OPT fractional can satisfy all 4.

Randomized rounding summary

important technique for obtaining approx algs for NP-hard problems

Recipe

1. Set problem up as integer linear program. vars $\in \{0,1\}$
2. Relax ILP $\longrightarrow$ LP vars $\in [0,1]$
3. Solve LP to optimality $\longrightarrow$ x_i^* $i=1,\dots,n$
4. Construct solution by randomly rounding vars $\longrightarrow \{0,1\}$
   treat x_i^* as probability
5. bound quality of soln by comparing to LP OPT

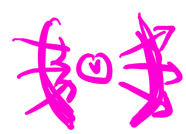
Another example: congestion minimization

Input: directed graph $G=(V,E)$
set of pairs (s_i,t_i) $i=1..k$

Output: path P_i from $s_i \rightsquigarrow t_i$ $\forall i=1..k$
s.t. $\underbrace{\text{congestion}}_{\boxed{\text{max \# paths that intersect any edge}}}$ is minimized

integer multicommodity flow

NP-hard



Approx alg via randomized rounding

① Set up ILP [multi commodity flow]

vars $f_i(e) \in \{0,1\}$ flow from $s_i \rightsquigarrow t_i$ on edge e

C congestion

$$\min \; C$$

subject to

$$\sum_{\substack{e \text{ s.t.} \\ e=(u\to v) \\ \text{for some } u}} f_i(e) = \sum_{\substack{e \text{ s.t.} \\ e=(v\to w) \\ \text{for some } w}} f_i(e) \qquad \forall v \neq s_i, t_i \quad \forall i$$

conservation of flow

$$\sum_{\substack{e \text{ s.t.} \\ e=(s_i\to u) \\ \text{for some } u}} f_i(e) = 1 \qquad \forall i$$

route 1 unit of flow from $s_i \rightarrow t_i$

$$\sum_i f_i(e) \le C \qquad \forall e$$

congestion bound

$$f_i(e) \in \{0,1\} \qquad \forall i,e$$

km

② Relax to LP

$$0 \le f_i(e) \le 1 \qquad \forall i,e.$$

③ Solve LPR $\longrightarrow$ $\boxed{f_i^*(e)}$ $\forall i,e$, $\boxed{C^*}$

$$\boxed{C^* \le OPT}$$

④ Round soln

Decompose flows into paths.

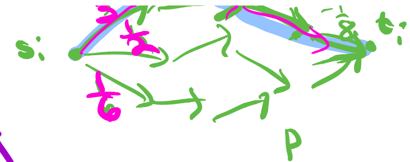
$\mathcal{P}_i = \{P \mid P \text{ is a path from } s_i \text{ to } t_i\}$

for each i, find set $\mathcal{P}_i^* \subseteq \mathcal{P}_i$ & f_P^i

s.t.

$$\sum_{\substack{P \in \mathcal{P}_i^* \\ \text{s.t. } \boxed{e \in P}}} f_P^i = f_i^*(e) \quad \leftarrow \forall \text{ edge } e.$$

$$\sum_{P \in \mathcal{P}_i^*} f_P^i = 1$$



f_P^i fraction on path P.

pick path $P \in \mathcal{P}_i$ w.p. $\boxed{f_P^i}$

$\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{6}$

P_i selected for routing $s_i \rightsquigarrow t_i$

⑤ Bound result.

Theorem If $C^* \geq c \ln n$ for some const c, then whp the max congestion on any edge $\leq C^* + \sqrt{cC^* \ln n}$

$\Rightarrow$ approx ratio ≤ 2

Proof Fix $e = (u \to v)$

Let $X_i(e) = \begin{cases} 1 & \text{if } e \in P_i \\ 0 & \text{o.w.} \end{cases}$

$E(X_i(e)) = \sum_{\substack{P \in \mathcal{P}_i \text{ s.t.} \\ e \in P}} f_P^i = f_i^*(e)$

congestion on edge e in rounded solution.

$X(e) = \sum_{i=1}^{k} X_i(e) \quad \forall e \quad E(X(e)) = \sum_{i=1}^{k} f_i^*(e) \leq C^*$

$$\Pr\left(X(e) \geq C^* + \sqrt{cC^* \ln n}\right) = \Pr\left(X(e) \geq (1+\delta)\underbrace{E(X(e))}_{\leq C^*}\right)$$

$0 \leq \sqrt{\frac{c \ln n}{C^*}} \leq 1$

$$\Pr(X \geq (1+\delta)\mu) \leq e^{-\delta^2 \mu/3} \qquad 0 \leq \delta \leq 1$$

$$\leq e^{-\frac{c \ln n}{C^*} C^*/3}$$

$$= e^{-c \ln n/3}$$

$$= n^{-c/3}$$

$\Pr(\exists e \text{ s.t. } X(e) \geq C^* + \sqrt{cC^* \ln n})$

$\leq \underset{\#\text{edges}}{n^2}\, n^{-c/3} \leq \frac{1}{n^2}$ $\quad c \geq 12$

If only know $C^* \geq 1$ $\qquad\qquad (1+\delta) = \frac{\ln n}{\ln \ln n}$

Theorem $\Pr\left(\text{any edge has congestion} \geq \boxed{\frac{6 \ln n}{\ln \ln n}}\, C^*\right) \leq \frac{1}{n}$

$\uparrow \alpha$

gives approx ratio of α

Proof Fix $e = (u \to v)$

Let $X_i(e) = \begin{cases} 1 & \text{if } e \in P_i \\ 0 & \text{o.w.} \end{cases}$

$$E(X_i(e)) = \sum_{\substack{P \in \mathcal{P}_i \text{ s.t.} \\ e \in P}} f_p^i = f_i^*(e)$$

Let $X(e) = \sum_{i=1}^{k} X_i(e) \implies \forall e \quad E(X(e)) = \sum_{i=1}^{k} f_i^*(e) \leq C^*$

$$\Pr\left(X(e) \geq \underbrace{(1+\delta)}_{\alpha} C^*\right) \leq e^{-C^*[(1+\delta)\ln(1+\delta) - \delta]} \qquad \text{[Chernoff bound]}$$

$$\Pr\left(X(e) \geq \alpha C^*\right) \leq e^{-C^*[\alpha \ln \alpha + 1 - \alpha]} \underset{C^* \geq 1}{\leq} e^{-\alpha(\ln \alpha - 1)}$$

$\alpha(\ln \alpha - 1)$

$= \frac{6 \ln n}{\ln \ln n}\left[\ln \ln n + \ln 6 - \ln \ln \ln n - 1\right]$

$\geq 3 \ln n$

$$\leq e^{-3 \ln n} = \frac{1}{n^3}$$

$$\implies \Pr\left(\text{any edge has congestion} \geq \alpha C^*\right) \underset{\text{union bound}}{\leq} \sum_e \Pr(X(e) \geq \alpha C^*)$$

$$\leq n^2 \cdot \frac{1}{n^3} = \frac{1}{n}$$

Optimal!

$\exists$ graphs with $\frac{OPT}{C^*} = \Omega\left(\frac{\ln n}{\ln \ln n}\right)$ integrality gap

so can't do better with such an approach

Hardness (directed graphs)

Every poly time alg has $\Omega\left(\frac{\log n}{\log \log n}\right)$ approx ratio

unless $NP \subseteq BPTIME\left(n^{O(\log \log n)}\right)$