Coloring

$G=(V,E)$ graph

graph is k-colorable if $\exists$ assignment $f: V \to [k]$

s.t. $f(i) \neq f(j)$ $\forall$ $(i,j) \in E$

$\chi(G)$:
chromatic # of G: smallest k s.t. G k-colorable

SDP rounding approach for k-colorability

- need to divide vertices into k sets s.t. every edge cut
- approach: map vertices to unit vectors in $\mathbb{R}^n$
maximize distances between endpts of edges
- use randomized procedure to partition vertices into k sets

$$\boxed{\begin{array}{ll} \min\ t & \\ v_i \cdot v_j \le t & \forall (i,j) \in E \\ v_i \cdot v_i = 1 & \forall i \in V \\ v_i \in \mathbb{R}^n & \forall i \end{array}}$$

relationship to colorability?
not obviously a relaxation

Lemma

Let t^* be optimal value of SDP

① If G is k-colorable, then $t^* \le -\frac{1}{k-1}$

② If G has a k-clique, then $t^* \ge -\frac{1}{k-1}$

Proof:

① Can explicitly construct k vectors $w_1, \dots, w_k$ s.t. $w_i \cdot w_j = \begin{cases} -\frac{1}{k-1} & i \ne j \\ 1 & i = j \end{cases}$
i.e. feasible soln $\quad w_i \in \mathbb{R}^k$

Construct by induction on k

$k=2$ ✓

$k \longrightarrow k+1$

$w_1', \dots, w_k'$ satisfy IH

$w_{k+1} = (0, \dots 0, 1)$

$w_i = \left(\sqrt{1-\frac{1}{k^2}}\, w_i', -\frac{1}{k}\right)$

$w_i \cdot w_i = \left(1-\frac{1}{k^2}\right) w_i' \cdot w_i' + \frac{1}{k^2} = 1$ ✓

$w_i \cdot w_j = \left(1-\frac{1}{k^2}\right) w_i' \cdot w_j' + \frac{1}{k^2}$

$\leq \left(1-\frac{1}{k^2}\right)\left(\frac{-1}{k-1}\right) + \frac{1}{k^2}$

$= \frac{(k^2-1)}{k^2}\left(\frac{-1}{k-1}\right) + \frac{1}{k^2} = -\frac{k+1}{k^2} + \frac{1}{k^2} = -\frac{1}{k}$ ✓

$w_i \cdot w_k = -\frac{1}{k}$ ✓

(2) $i=1 \dots k$ is clique $\Rightarrow \forall\, 1 \leq i < j \leq k \quad v_i \cdot v_j \leq t^*$

claim $\boxed{\frac{1}{k(k-1)} \sum_{1 \leq i,j \leq k} (v_i \cdot v_j)} \geq -\frac{1}{k-1}$ (*)

$\bar{a}$

$\Rightarrow \exists\, i, j$ s.t. $(v_i \cdot v_j) \geq -\frac{1}{k-1}$

$\Rightarrow t^* \geq -\frac{1}{k-1}$

to prove (*) observe

$\left(\sum_{i=1}^{k} v_i, \sum_{i=1}^{k} v_i\right) = \sum_{i=1}^{k} (v_i, v_i) + k(k-1)\bar{a} = k + k(k-1)\bar{a} \geq 0 \quad \Rightarrow \bar{a} \geq -\frac{1}{k-1}$

Define $\Theta(G) = 1 - \frac{1}{t^*}$ Lovasz Theta Fn

$t^* = -\frac{1}{k-1} \implies \Theta(G) = k$ ("graph is vector k-colorable")

(indep # of $\bar{G}$) $\omega(G) \leq \Theta(G) \leq \chi(G)$ (size of clique cover of $\bar{G}$)

clique # — size of largest clique.

chromatic #

A graph is perfect if $\omega(G') = \chi(G')$ for all vertex-induced subgraphs

(possible to have poly separation)

$\boxed{\text{Using this to color 3-colorable graph}}$

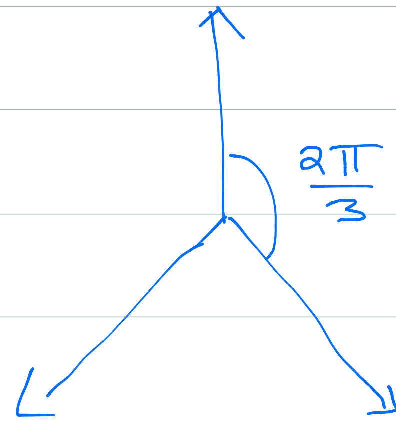
$\min \lambda$

$v_i \cdot v_j \le \lambda \quad \forall (i,j) \in E$

$v_i \cdot v_i = 1 \quad \forall i \in V$

$v_i \in \mathbb{R}^n$

$\boxed{\text{3 colorable} \Rightarrow \lambda^* \le -\tfrac{1}{2}}$



We'll show how to find coloring with $\tilde{O}(n^{0.387})$ colors [Karger, Motwani, Sudan]

current best $O(n^{0.211})$ [Arora, Chlamtac, Charikar]

Algorithm:

solve SDP

choose $t = 2 + \log_3 \Delta$ random hyperplanes thru origin
$\Rightarrow$ partition vectors into 2^t subsets
color vectors in each region w/ diff color

} repeat until properly colored

uses $2^t = 4 \cdot 2^{\log_3 \Delta} = 4\Delta^{\log_3 2}$ colors

Claim: produces $\underbrace{\text{semi-coloring}}$ w.p. $\geq \frac{1}{2}$ (*)

coloring of nodes s.t.
$\leq \frac{n}{4}$ edges have same color at both endpts

$\Rightarrow$ at least $\frac{n}{2}$ vertices properly colored.
(edges between them

k colors sufficient to get semi coloring

$\Rightarrow$ graph can be properly colored w/ $O(k \log n)$ colors

Proof of (*)

Fix $(i,j) \in E$

arccos $\downarrow$ fn

$$\Pr(i \,\&\, j \text{ get same color}) = \left(1 - \frac{\arccos(v_i \cdot v_j)}{\pi}\right)^t \leq \left(1 - \frac{\arccos(\lambda^*)}{\pi}\right)^t$$

$$\leq \left(1 - \frac{\arccos(-\frac{1}{2})}{\pi}\right)^t$$

$$\leq \left(1 - \frac{1}{\pi}\frac{2\pi}{3}\right)^t = \frac{1}{3^t} \leq \frac{1}{9\Delta}$$

$$\Rightarrow E(\#\text{ edges with same color}) \leq \frac{|E|}{9\Delta} \leq \frac{n\Delta}{2 \cdot 9\Delta} = \frac{n}{18}$$

$\Rightarrow$ By Markov Ineq $\Pr\left(\#\underset{\text{same color}}{\text{edges w/}} \geq \frac{n}{4}\right) \leq \frac{n/18}{n/4} < \frac{1}{2}$

$\Rightarrow$ w.p. $\geq \frac{1}{2}$ get semicoloring using $k = 4\Delta^{\log_3 2}$ colors

$\Rightarrow$ get coloring using $O(\log n \, \Delta^{\log_3 2})$ colors

if $\Delta = n \Rightarrow \tilde{O}(n^{\log_3 2}) = \tilde{O}(n^{0.631})$ colors

☹ $\exists$ simple alg that uses $O(\sqrt{n})$ colors to color 3-colorable graph [Wigderson]

Uses 2 key facts:

- neighborhood of vertex 2-colorable
- graph w/ max degree Δ can be colored w/ $\Delta+1$ colors

while $\exists$ node of deg $\geq \sqrt{n}$, find it & 2-color its neighborhood using fresh colors

can do this $\leq \sqrt{n}$ times

at end max degree $< \sqrt{n}$

color using $\sqrt{n}$ additional colors

Can use this to improve previous alg.

Let Δ^* be parameter

1. Pick a vertex of deg $\geq \Delta^*$ & 3-color it & neighbors
2. Repeat step 1 until all vertices have deg $< \Delta^*$

$\Big\}\ \leq 3\frac{n}{\Delta^*}$ colors

3. Run SDP based coloring alg to color rest $\Big\}\ \tilde{O}\left(\Delta^{*\,\log_3 2}\right)$ colors

choose Δ^* to minimize $\frac{3n}{\Delta^*} + (\Delta^*)^{\log_3 2}$

$\Rightarrow \Delta^* = n^{1/1.63}$

$\Rightarrow \tilde{O}(n^{0.39})$

Wide open

Is there an alg for 3-coloring $\underbrace{\text{, that}}_{\text{a 3-colorable graph}}$ uses polylog n colors?