Random walks on undirected graphs

$G=(V,E)$ undirected graph

consider simple random walk on this graph:

$p_{ij} = \frac{1}{d_i} \quad \forall (i,j) \in E$ [MC is periodic iff graph is bipartite]

$\boxed{\pi_i = \frac{d_i}{2m} \qquad m: \#\text{ of edges}}$

Some key quantities:

hitting time	$h_{ij} = E(T_{ij})$
commute time	$c_{ij} = h_{ij} + h_{ji}$
covertime	$C(G)$ = time to visit all vertices

$$h_{ii} = \frac{1}{\pi_i} = \frac{2m}{d_i}$$

Lemma $\forall$ edge (i,j) $\quad h_{ij} + h_{ji} \le 2m$

Proof Consider corresponding random walk on directed edges - $2m$ states

$$q_{(i,j)(j,k)} = p_{jk} = \frac{1}{d_j}$$

Claim: Q doubly stochastic (both rows & cols sum to 1

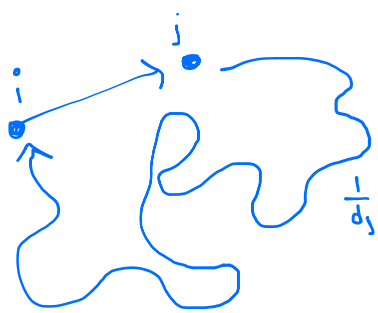
$$\sum_{i \text{ s.t. } (i,j) \in E} q_{(i,j)(j,k)} = \sum_{\substack{i \text{ s.t.} \\ (i,j) \in E}} \frac{1}{d_j} = \frac{d_j}{d_j} = 1$$

Fact: stationary dist'n of any doubly stochastic MC is uniform

$$\Rightarrow \text{for } Q \quad \pi_{(i,j)} = \frac{1}{2m}$$

$$\Rightarrow h_{(i,j)(i,j)} = 2m$$

$$h_{ij} + h_{ji} \leq h_{(i,j),(i,j)}$$



$\frac{1}{d_j} \cdots \cdots \frac{1}{d_i}$

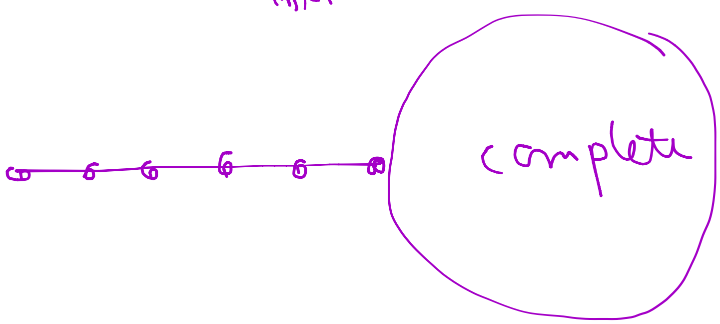
$C_u(G)$: covertime starting at u

$C(G) = \max_u C_u(G)$

Thm $E(C(G)) \le 2m(n-1)$

Construct spanning tree

$E(C(G)) \le \sum_{(i,j)\in T} (h_{ij} + h_{ji}) \le 2m(n-1)$



lollipop
$C(G) = \Theta(n^3)$

line
$C(G) = \Theta(n^2)$

complete graph

Application: s-t connectivity

Given G undirected $s, t \in V$

decide whether s & t are in same CC.

DFS $\left\{ \begin{array}{l} O(m) \text{ time} \\ O(n) \text{ space} \end{array} \right.$ keep track of all vertices search has visited so far
BFS

$\boxed{\text{Observation:}}$ very simple randomized alg using log space (workspace)

input on separate read-only tape

simulate r.w. of length $2n^3$ on G starting from s

$\Pr(\text{doesn't reach } t \text{ when } \exists \text{ path}) \leq \frac{1}{2}$

RW on regular graph degree d

$$P_{ij} = \begin{cases} \frac{1}{d} & (i,j) \in E \\ 0 & \text{o.w} \end{cases}$$

$$\pi = \left(\frac{1}{n}, \cdots, \frac{1}{n}\right)$$

Brief detour on $\boxed{\text{mixing time}}$

how long it takes to converge to π

related to algebraic properties of P

suppose $q^{(0)}$ initial distn & $q^{(t)}$ distn after t steps

$$q^{(t)} = q^{(0)} P^{(t)}$$

How close is $q^{(t)}$ to π?

Spectral Thm

If $M \in \mathbb{R}^{n \times n}$ symmetric, then

- all n eigenvalues are real (solns of $\det(A - \lambda I) = 0$)
- $\exists$ orthonormal set of eigenvectors $v_1, \ldots, v_n$ corresponding to eigenvalues $\quad v_i \cdot v_j = \begin{cases} 1 & i = j \\ 0 & \text{o.w.} \end{cases}$

$$Mv_i = \lambda v_i$$

$$\Rightarrow \quad M = \sum_{i=1}^{n} \lambda_i v_i v_i^T$$

$$= \underbrace{\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}}_{\Phi^T} \underbrace{\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} — & v_1 & — \\ — & v_2 & — \\ & \vdots & \\ — & v_n & — \end{pmatrix}}_{\Phi}$$

$$\Phi \Phi^T = \Phi^T \Phi = I$$

$$\Rightarrow M^t = \sum_i \lambda_i^t v_i v_i^T \qquad (*)$$

$\boxed{\text{Perron-Frobenius Thm}}$

If $A > 0$ and $A^m \gg 0 \quad \forall\ m \geq M$

- $\exists\ \vec{x} \gg 0$ s.t. $Ax = \lambda^* x$
- If $\lambda \neq \lambda^*$ is any other eigenvalue of A

then $|\lambda| < \lambda^*$

P lazy random walk

$$p_{ij} \begin{cases} \frac{1}{2} & j = i \\ \frac{1}{2d} & (i,j) \in E \\ 0 & \text{o.w.} \end{cases} \qquad 0 \leq \text{eigenvalues} \leq 1$$

Back to mixing time

$$P^t = \underbrace{v_1 v_1^T}_{\frac{1}{n} J} + \sum_{i \geq 2} \lambda_i^t v_i v_i^T \qquad v_1^T = \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right)$$

all 1's matrix

$$q^{(0)} = \sum_i c_i v_i^T$$

where $c_i := q^{(0)} \cdot v_i$

$$q^{(0)} P^t = \sum_i c_i v_i^T \sum_j \lambda_j^t v_j v_j^T$$

$$= \sum_i c_i \lambda_i^t v_i^T$$

$$c_1 = q^{(0)} \cdot v_1 = \frac{1}{\sqrt{n}} \sum q_i^{(0)} = \frac{1}{\sqrt{n}}$$

$$\lambda_1 = 1$$

$$= \underbrace{\frac{1}{\sqrt{n}} v_1^T}_{= \left(\frac{1}{n}, \dots, \frac{1}{n}\right) = \pi} + \sum_{i \geq 2} c_i \lambda_i^t v_i^T$$

$$\left\| q^{(0)} P^t - \pi \right\| = \left\| \sum_{i=2}^n c_i \lambda_i^t v_i^T \right\|$$

$$= \sqrt{\sum_{i=2}^n c_i^2 \lambda_i^{2t}}$$

$$\leq \lambda_2^t \sqrt{\sum_i c_i^2}$$

$$\sqrt{\sum_i c_i^2} = \|q^0\| \leq 1$$

$t = \Omega\left(\frac{1}{(1-\lambda_2)} \log n\right) \Rightarrow$ above n^{-c}

takes $O\left(\frac{1}{(1-\lambda_2)} \log n\right)$ steps to converge to stationary distn

$1-\lambda_2$ big $\equiv \lambda_2 \ll 1 \Rightarrow$ fast convergence d-regular graphs with $\lambda_2 << 1$ called $\boxed{\text{expanders}}$

sets have const edge expansion

- Random d-reg. graphs - even d small const are expanders

  $\lambda_2 = \Theta\left(\frac{1}{\sqrt{d}}\right) \Rightarrow$ mixing time $O(\log n)$

- $\exists$ explicit constructions

- wide variety of uses in TCS

  complexity theory

  design of robust computer networks

  error correcting codes

  pseudorandomness