"Puzzle"

$A \subseteq \{0, 1, 2, \ldots, p-1\}$ $\quad |A| \geq \frac{p}{2}$

Show $\exists$ int n s.t. all gaps of $\{nA \bmod p\} \leq 5\sqrt{p}$

First Moment Method

r.v. X takes some value $\geq E(X)$
$\leq E(X)$

$X \geq 0$ int valued $\Rightarrow Pr(X \leq 1) \leq E(X)$
$\Rightarrow X = 0$ w.h.p. if $E(X) \ll 1$

Proving $P(X \geq 1)$ large?

$E(X)$ large insufficient – could be HUGE w/ negligible prob.

2nd Moment Method

pretty much any use of 2nd moment $E(X^2)$

Chebychev's Inequality

$\forall \lambda > 0$

$$Pr(|X - \mu| \geq \lambda\sigma) \leq \frac{1}{\lambda^2}$$

X nonnegative integer-valued r.v.

Another version

$$\Pr(X=0) \le \frac{Var(X)}{E(X^2)}$$

Pf:

$$\Pr(X=0) \le \Pr(|X-\mu| \ge \mu) \le \frac{\sigma^2}{\mu^2} = \frac{Var(X)}{(E(X))^2}$$

Corollary

If $Var(X) = o(E(X)^2)$ then $\Pr(X>0) = 1-o(1)$

Typically $X = X_1 + X_2 + \dots + X_n$

Claim: $Var(X) = \sum_{i=1}^{n} Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j)$

$Cov(X,Y) = E(XY) - E(X)E(Y)$

Observe $Cov(X,Y) = 0$ if X & Y indep

$$Var(X+Y) = E\left[\left(X+Y-E(X+Y)\right)^2\right]$$

$$= E\left[(X-E(X))^2 + (Y-E(Y))^2 + 2(X-E(X))(Y-E(Y))\right]$$

$$= Var(X) + Var(Y) + 2\,Cov(X,Y)$$

Often X_i's are indicator r.v.'s

$$X_i = \begin{cases} 1 & p_i \\ 0 & 1-p_i \end{cases}$$

$$Var(X_i) = p_i(1-p_i) \leq p_i = E(X_i)$$

$$\Rightarrow Var(X) \leq E(X) + \sum_{i \neq j} Cov(X_i, X_j)$$

Example 1: Consider $G_{n,p}$ graph on n vertices where each edge present w/ prob p (independently)

For what p does $G_{n,p}$ have a clique with 4 or more vertices?

Claim: ① almost surely doesn't for $p = o(n^{-2/3})$

② almost surely does for $p = \omega(n^{-2/3})$

Proof of ①

$\binom{n}{4}$ possible 4-cliques $\qquad C_1, \dots, C_{\binom{n}{4}}$ enumeration

$$X_i := \begin{cases} 1 & C_i \text{ clique} \\ 0 & \text{o.w.} \end{cases}$$

$$X = \sum_{i=1}^{\binom{n}{4}} X_i \quad \text{\# of 4-cliques}$$

$$E(X) = \binom{n}{4} p^6 \simeq \frac{n^4}{4!} p^6$$

for $p = o(n^{-2/3})$ $\qquad E(X) = o(1)$

$X \geq 0$, integer $\Rightarrow \Pr(X \geq 1) \leq E(X) = o(1)$ $\qquad$ first moment method

Proof of 2

For $p = \omega(n^{-2/3})$ $\quad E(X) \xrightarrow[n\to\infty]{} \infty$ $\quad$ Insufficient by itself

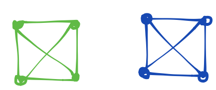
So we use 2nd moment method, we'll show $Var(X) = o(E(X)^2)$

need to compute $Var(X)$

$Cov(X_i, X_j)$

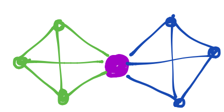
type 1

$C_i \cap C_j = \phi$ $\quad Cov(X_i, X_j) = 0$



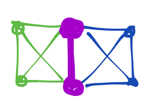
type 2

$|C_i \cap C_j| = 1$ $\quad Cov(X_i, X_j) = 0$



type 3

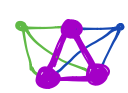
$|C_i \cap C_j| = 2$



$Cov(X_i, X_j) \leq E(X_i X_j) = p^{11}$

type 4

$|C_i \cap C_j| = 3$



$Cov(X_i, X_j) \leq E(X_i X_j) = p^9$

$$\mathrm{Var}(X) \leq E(X) + \sum_{\substack{1 \leq i,j \leq m \\ i \neq j}} \mathrm{Cov}(X_i, X_j)$$

$$\leq \binom{n}{4} p^6 + \underbrace{\binom{n}{6}\binom{6}{2;2;2} p^{11}}_{\text{type 3}} + \underbrace{\binom{n}{5}\binom{5}{3;1;1} p^9}_{\text{type 4}}$$

$$\leq c\left(n^4 p^6 + n^6 p^{11} + n^5 p^9\right) = o\left([E(X)]^2\right)$$

since

$$[E(X)]^2 = \left(\binom{n}{4} p^6\right)^2 = \Theta\left(n^8 p^{12}\right)$$

$$p = \omega\left(n^{-2/3}\right)$$

$$\Rightarrow \Pr(X=0) \leq \frac{\mathrm{Var}(X)}{[E(X)]^2} = o(1)$$

Evolution of Random graphs

The following statements hold almost surely $\equiv$ with prob $\to 1$ as $n \to \infty$

$p = o\left(\frac{1}{n}\right)$

forest
no cycles

$c < 1$

$p \leq \frac{c}{n}$

largest connected component has size $\Theta(\log n)$

$p = \frac{1}{n}$

largest CC has size $\Theta(n^{2/3})$

$c > 1$

$p \geq \frac{c}{n}$

largest CC
has size $\Theta(n)$
"giant component"

$p = \frac{\ln n}{n}$

graph is connected

has Hamiltonian cycle

contains matching

...

i

Graph properties expressible in "First-Order theory" of graphs

language: vars $(x, y, z, \dots)$ represent vertices

equality $x = y$; adjacency $x \sim y$

$\wedge, \vee, \neg$ $\forall, \exists$

Examples:

G contains triangle: $\exists x \exists y \exists z \;\; x \sim y \wedge y \sim z \wedge z \sim x$

no isolated pt $\forall x \, \exists y \, (x \sim y)$

diameter 2 $\forall x \, \forall y \, \exists z \, \big((x = y) \vee (x \sim y) \vee ((x \sim z) \wedge (z \sim y)) \big)$

(G Hamiltonian, G connected) not FO properties

For A a FO property, let

$$\Pr(G(n,p) \models A) \equiv \Pr\left(\text{random graph from } G(n,p) \text{ has property } A\right)$$

$\boxed{\text{Thm [Fagin] [Glebskii, Kogan, Liagonkii, Talanov]}}$

$\forall$ fixed p, $0<p<1$, and any FO graph property

$$\lim_{n\to\infty} \Pr(G(n,p) \models A) = 0 \text{ or } 1$$

$\boxed{\text{Thm [Shelah, Spencer]}}$

$\forall$ irrational α, $0<\alpha<1$, setting $p=p(n)=n^{-\alpha}$ and for any FO property A

$$\lim_{n\to\infty} \Pr(G(n,p) \models A) = 0 \quad \text{or} \quad 1$$

this is about threshold fns

What happens at $n^{-\pi/7}$?

Nothing!!

Ex $p=n^{-2/3}$ threshold fn for K_4

$p \ll n^{-2/3}$ no K_4

$p=n^{-2/3}$ $\Pr(K_4)=1-e^{-1/24}$

$p \gg n^{-2/3}$ K_4

Another useful 2nd moment inequality

$$\boxed{\Pr(X>0) \geq \frac{(E(X))^2}{E(X^2)}}$$

Follows from Cauchy-Schwartz

$$\boxed{[E(XY)]^2 \leq E(X^2)\,E(Y^2)}$$

Proof:

wlog $E(X^2)>0$, $E(Y^2)>0$

Let $U = \frac{X}{\sqrt{E(X^2)}}$ $V = \frac{Y}{\sqrt{E(Y^2)}}$

$2|UV| \leq U^2+V^2$

$\Rightarrow 2\,|E(UV)| \leq 2\,E(|UV|)$

$\leq E(U^2) + E(V^2) = 2$

$\Rightarrow [E(UV)]^2 \leq 1$

$\equiv [E(XY)]^2 \leq E(X^2)E(Y^2)$

Set $Y = 1_{X>0}$

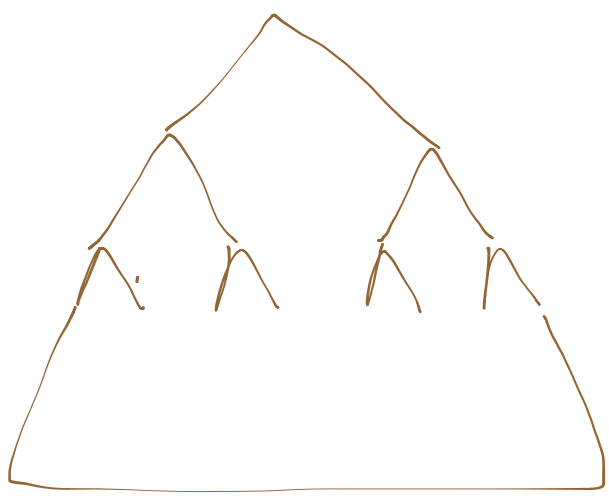
$$[E(X)]^2 \leq E(X^2)\underbrace{E([1_{X>0}]^2)}_{\Pr(X>0)}$$

Percolation on Tree

Complete binary tree of depth n

each edge present w/ prob p

Is there a surviving path from root to a leaf?



X # leaves reachable from root

$E(X) = ?$

$X = X_1 + X_2 + \dots + X_{2^n}$

X_i indicates if i^{th} leaf is reachable

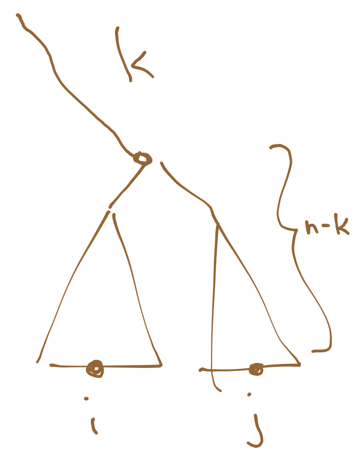
$$E(X) = 2^n \cdot p^n = \begin{cases} \to \infty & p > \frac{1}{2} \\ = 1 & p = \frac{1}{2} \\ \to 0 & p < \frac{1}{2} \end{cases}$$

What can we say about $\Pr(X > 0)$?

$p > \frac{1}{2}$

$$\Pr(X>0) \geq \frac{[E(X)]^2}{E(X^2)}$$

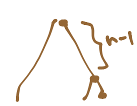
$$X^2 = \left(\sum X_i\right)^2 = \sum_i X_i^2 + \sum_i \sum_{j \neq i} X_i X_j$$



$$E(X_i X_j) = p^{2n-k}$$

$$\sum_i \sum_{j \neq i} E(X_i X_j) = \sum_{k=0}^{n-1} 2^k \, 2^{n-k} \, 2^{n-k-1} \, p^{2n-k}$$

$$= \frac{1}{2} \sum_{k=0}^{n} 2^{2n} \frac{1}{2^k} p^{2n} \frac{1}{p^k} = \frac{1}{2} (2p)^{2n} \underbrace{\sum_{k=0}^{n-1} \frac{1}{(2p)^k}}_{\frac{1 - \frac{1}{(2p)^n}}{1 - \frac{1}{2p}}}$$



$p > \frac{1}{2}$

$2p > 1$

$$\leq \frac{1}{2} (2p)^{2n} \frac{2p}{2p-1} = \frac{p}{2p-1} (2p)^{2n}$$

$$E(X^2) \le \left(\frac{p}{2p-1}\right)(2p)^{2n}\left[1+o(1)\right]$$

$$\Pr(X>0) \ge \frac{[E(X)]^2}{E(X^2)} \ge \frac{2p-1}{p}(1-o(1)) > c \qquad \text{for } p > \tfrac{1}{2}$$

$p = \frac{1}{2}$

$$\sum_i \sum_{j \ne i} E(X_i X_j) = \sum_{k=0}^{n-1} 2^k \, 2^{n-k} \, 2^{n-k-1} \, p^{2n-k}$$

$$= \frac{n}{2}$$

$$E(X^2) = 1 + \frac{n}{2} \qquad [E(X)]^2 = 1$$

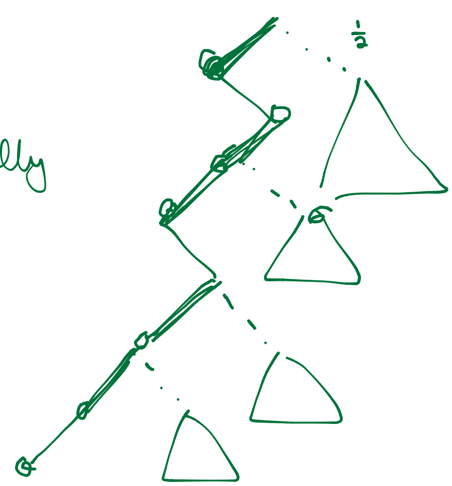
$$\Pr(X>0) \ge \frac{(E(X))^2}{E(X^2)} \ge \frac{2}{n+2}$$

Other direction : L event that a vertex with at least $\frac{n}{2}$ left turns survives

$$E(X \mid L) \ge \frac{n}{4}$$

toss coins to produce lexicographically leftmost path

$\frac{n}{2}$ left turns for each right child exp



$$\Pr(L) \le \frac{E(X)}{E(X \mid L)} \le \frac{4}{n} \qquad \Pr(X > 0) \le \Pr(L) + \Pr(R) \le \frac{8}{n}$$