Warmup: toss fair coin

Exp # steps to see H?

HT?

HH?

HHH HTH?

$$X = \overset{H}{2} + 1 + \tfrac{1}{2}\overset{\dots HT}{\overbrace{X}} + \tfrac{1}{2}\left[\overset{HHH}{1 + \tfrac{1}{2}X}\right]$$

$$= 3\tfrac{1}{2} + \tfrac{3}{4}X$$

$$\tfrac{1}{4}X = \tfrac{7}{2} \Rightarrow X = 14$$

Organized approach (in terms of fair gambling):

each step + a new gambler arrives

bets $1 on first H

$\frac{1}{2}$ Loses $\frac{1}{2}$ Wins gets $2

bets it on T

$\frac{1}{2}$ Loses $\frac{1}{2}$ Wins gets $4

bets it on H

$\frac{1}{2}$ Loses $\frac{1}{2}$ wins gets $8

$$T\,H\,H\,T\,T\,\underbrace{H\,T\,\overbrace{H}^{2}}_{8}$$

every gambler bets fair bets. at end, say after n coin tosses get HTH

Net profit of all gamblers $= 8 + 2 - n$

$$E(\text{net profit}) = E(8+2-N) = 0$$

all bets are fair

$$\Rightarrow E(N) = 10$$

HTH	10
HHH	14

A stochastic process $\{X_t\}$ is a $\boxed{\text{martingale}}$ if $\forall i$

$$E(X_{i+1} \mid X_0 \ldots X_i) = X_i$$

$E(Y|Z)$ is a r.v.
that takes value $E(Y|Z=z)$
with prob $Pr(Z=z)$

Example: gambler walks into casino with $\$X_0$

- X_i amt of money gambler has after i games
- every game is "fair" i.e. expected winnings $= 0$

$\{X_t\}$ is $\boxed{\text{martingale}}$ with respect to $\{Y_t\}$ where $X_t = f(Y_0, .., Y_t)$ if $\forall i$

$$E(X_{i+1} \mid Y_0, .., Y_i) = X_i$$

Some common martingales:

① Sums of indep random variables

$Y_0 = 0 \quad Y_1, \ldots, Y_n$ iid w/ $E(Y_k) = 0 \quad \forall k$

Define $X_0 = 0$, $X_n = Y_1 + \cdots + Y_n$

$\{X_t\}$ is a martingale wrt $\{Y_t\}$

$$E(X_{n+1} \mid Y_0, \ldots, Y_n) = E(X_n + Y_{n+1} \mid Y_0, \ldots, Y_n)$$

$$= E(X_n \mid Y_0, \ldots, Y_n) + E(Y_{n+1} \mid Y_0, \ldots, Y_n)$$

$$= X_n + E(Y_{n+1})$$

$$= X_n$$

② Variance of a sum

$Y_0 = 0 \quad Y_1, \ldots, Y_n$ iid w/ $E(Y_k) = 0 \quad \forall k \qquad E(Y_k^2) = \sigma^2$

Define $X_0 = 0 \qquad X_n = \left(\sum_{k=1}^{n} Y_k\right)^2 - n\sigma^2$

$\{X_t\}$ is martingale w.r.t. $\{Y_t\}$

$$E(Y_{n+1} \mid Y_0,\dots,Y_n) = E\Big[\big(Y_{n+1} + \sum_{k=1}^{n} Y_k\big)^2 - (n+1)\sigma^2 \;\Big|\; Y_0,\dots,Y_n\Big]$$

$$= E\Big(Y_{n+1}^2 + 2Y_{n+1}\big(\sum_1^n Y_k\big) + \big(\sum_{k=1}^{n} Y_k\big)^2 - (n+1)\sigma^2 \;\Big|\; Y_0,\dots,Y_n\Big]$$

$$= X_n + \underbrace{E(Y_{n+1}^2)}_{\sigma^2} + 2\underbrace{E(Y_{n+1})}_{0}\big(\sum_{k=1}^{n} Y_k\big) - \sigma^2$$

$$= X_n$$

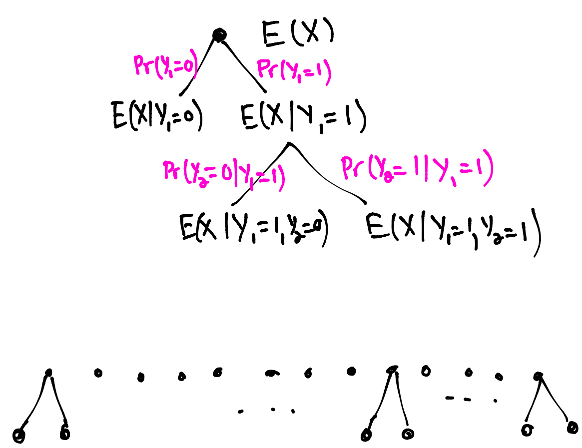
③ "Doob's" martingale process

$Y_1, Y_2, \dots$ arbitrary seq of random vars

X r.v. w/ finite expectation

$X_n = E(X \mid Y_1,\dots,Y_n)$ forms martingale wrt. $\{Y_n\}$ $\quad X_0 = E(X)$

$$X_0 = E(X)$$
$$X_1 = E(X \mid Y_1)$$
$$X_2 = E(X \mid Y_1, Y_2)$$
$$\vdots$$
$$X_n = E(X \mid Y_1,\dots,Y_n)$$
$$X_{n+1} = E(X \mid Y_1,\dots,Y_{n+1})$$



$\forall y_1 \dots y_n \quad E(X_{n+1} \mid Y_1 = y_1, \dots Y_n = y_n) = X_n$

$$E(V|W) = E\left[E[V|U,W]\,|\,W\right]$$

$$E(X_{n+1} | Y_1, \dots, Y_n)$$

$$= E\left(E(X | Y_1, \dots, Y_{n+1}) \,|\, Y_1, \dots, Y_n\right)$$

$$= E(X | Y_1, \dots, Y_n) = X_n$$

Ex: Edge exposure martingale

$G(n,p)$ random graph

label $m = \binom{n}{2}$ potential edges $e_1, \dots, e_m$

Let f be graph-theoretic fn e.g. chromatic #

max indep set size

$$Y_j \text{ ind. r.v.} = \begin{cases} 1 & \text{if edge } e_j \text{ present} \quad \text{prob } p \\ 0 & \text{otherwise} \end{cases}$$

$$X_k = E(f(G) | Y_1, \dots, Y_k)$$

$$X_0 = E(f(G)) \qquad X_m = f(G)$$

Some useful facts about martingales

$$\boxed{① \ E(X_n) = E(X_0)}$$

by induction

$$E(X_{n+1} \mid Y_0, \ldots, Y_n) = X_n$$

$$\underbrace{E\big(E(X_{n+1} \mid Y_0, \ldots, Y_n)\big)}_{= E(X_{n+1})} = E(X_n)$$

$$E(\underbrace{E(X \mid Y)}_{E(X \mid Y=y)}) = E(X)$$

with prob $Pr(Y=y)$

$$E(E(X \mid Y)) = \sum_y E(X \mid Y=y) \, Pr(Y=y)$$

$$= \sum_y \sum_x x \, Pr(X=x \mid Y=y) \, Pr(Y=y)$$

$$= \sum_x x \sum_y Pr(X=x, Y=y)$$

$$= \sum_x x \, Pr(X=x)$$

$\{Z_t\}$ martingale w.r.t. $\{X_t\}$

(2) For T a stopping time "know it when you see it"

$$E(Z_T) = E(Z_0)$$ optional sampling thm

whenever any of following hold

- Z_i's bounded $(\exists c \text{ s.t. } \forall i \ |Z_i| \leq c)$
- T is bounded
- $E(T) < \infty$ and $\exists c$ s.t. $E(|Z_{i+1} - Z_i| \mid X_1, \ldots, X_i) \leq c$

A r.v. T is called a "stopping time" w.r.t. $\{Y_n\}$ if

T takes values in $\{0, 1, 2, \ldots\}$

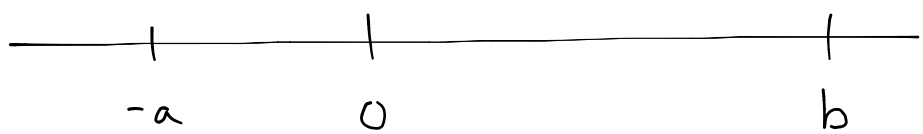
and if $\forall n \geq 0$, the event $\{T = n\}$ is determined by $Y_0, \ldots, Y_n$

i.e. can determine if $T = n$ or $T \neq n$ from knowledge of values $Y_0, \ldots, Y_n$

Applications of Optional Sampling Thm

① unbiased r.w. on line starting at 0

Pr(reach $-a$ before reaching b?)



$Y_i = \begin{cases} 1 & \text{w. prob } \frac{1}{2} \\ -1 & \frac{1}{2} \end{cases}$

$X_n = \sum_{i=1}^{n} Y_i$ martingale

$T = \min\{n \mid X_n = -a \text{ or } X_n = b\}$

T is a stopping time

Let $v_a = Pr(X_n$ reaches $-a$ before reaching $b)$

By O.S.T. $E(X_T) = E(X_0) = 0$

$$E(X_T) = v_a(-a) + (1-v_a)b = 0$$

$$\Rightarrow \quad v_a = \frac{b}{a+b}$$

② Same unbiased r.w. on line, same T

What is $E(T)$?

$Z_n = X_n^2 - n$ is a martingale

var of a sum $E(Y_i^2) = 1$

By O.S.T.

$E(Z_T) = E(Z_0) = 0$

$E(Z_T) = [v_a a^2 + (1-v_a)b^2] - E(T) \underset{\text{O.S.T.}}{=} 0$

$\Rightarrow E(T) = \frac{b}{a+b} a^2 + \frac{a}{b+a} b^2 = ab$

Same questions: biased r.w.

$$Y_i = \begin{cases} +1 & p \\ -1 & q \end{cases} \qquad p > q \ (= 1-p)$$

$$\left.\begin{aligned} X_n &= \sum_{i=1}^{n} Y_i - n(p-q) \\ X_n' &= \left(\frac{q}{p}\right)^{\sum_{i=1}^{n} Y_i} \qquad (X_0' = 1) \end{aligned}\right\} \text{martingales w.r.t. } \{Y_n\}$$

$$T = \min\left\{ n \,\middle|\, \sum_{i=1}^{n} Y_i = -a \text{ or } = b \right\}$$

$$v_a = \Pr\left(\sum_{i=1}^{n} Y_i \text{ reaches } -a \text{ before } b\right)$$

$$E(X_T') = E(X_0') = 1$$

$$E(X_T') = v_a \left(\frac{q}{p}\right)^{-a} + (1-v_a)\left(\frac{q}{p}\right)^{b}$$

$$\Rightarrow \quad v_a = \frac{1 - \left(\frac{q}{p}\right)^{b}}{\left(\frac{q}{p}\right)^{-a} - \left(\frac{q}{p}\right)^{b}}$$

③ Tail Inequalities (Large deviations)

$E(X_m) = E(X_0)$ how far can it be from its expectation?

Azuma-Hoeffding Inequality

$X_0, \dots, X_m$ martingale s.t. $\forall k$ $|X_k - X_{k-1}| \le c_k$

(c_k may depend on k)

Then $\forall t \ge 0$, any $R > 0$

$$\Pr(|X_t - X_0| > R) \le 2 e^{-\left[\frac{R^2}{2\sum_{k=1}^{t} c_k^2}\right]}$$

Proof:

By convexity of $f(x) = e^{\lambda x}$ we have $e^{\lambda x} \le \dfrac{(1-\frac{x}{c})e^{-\lambda c} + (1+\frac{x}{c})e^{\lambda c}}{2} = \ell(x)$
for $x \in [-c, c]$

$x = -c\,\frac{(1-\frac{x}{c})}{2} + c\,\frac{(1+\frac{x}{c})}{2}$

Thus if X has $E(X) = 0$ and $|X| \le c$ then

$$E(e^{\lambda X}) \le E(\ell(X)) = \frac{e^{\lambda c} + e^{-\lambda c}}{2} = \sum_{k=0}^{\infty} \frac{(\lambda c)^{2k}}{(2k)!} \le \sum_{k=0}^{\infty} \frac{(\lambda c)^{2k}}{2^k k!} = e^{(\lambda c)^2/2}$$

Therefore $E\left(e^{\lambda(X_{t+1} - X_t)} \middle| H_t\right) \le e^{(\lambda c_t)^2/2}$

$$\Rightarrow E\left(e^{\lambda X_{t+1}} \mid H_t\right) = e^{\lambda X_t} E\left(e^{\lambda(X_{t+1}-X_t)} \mid H_t\right) \le e^{\lambda X_t} e^{(\lambda c_t)^2/2}$$

Taking expectations $E\left(e^{\lambda X_{t+1}}\right) \le e^{(\lambda c_t)^2/2} E\left(e^{\lambda X_t}\right) \le e^{\lambda^2 \sum_{i=1}^{t+1} c_i^2/2}$

by induction on t

Finally, $P(X_t \ge R) = \Pr\left(e^{\lambda X_t} \ge e^{\lambda R}\right) \le e^{-\lambda R} e^{\lambda^2 \frac{\sum c_i^2}{2}}$

Optimizing, we choose $\lambda = \frac{R}{\sum_{i=1}^{t} c_i^2}$ $\Pr(X_t \ge R) \le e^{-R^2/2\sum_{i=1}^{t} c_i^2}$

$$-\left(\lambda R - \frac{\lambda^2 \sum c_i^2}{2}\right) = -\left(\frac{R^2}{\sum c_i^2} - \frac{R^2}{2\sum c_i^2}\right)$$

factor of 2 comes from $\Pr(X_t < -\lambda)$ obtained by replacing $X_t - X_{t-1}$ by $X_{t-1} - X_t$

Tail bounds r.w. on line $\quad Y_0 = 0 \quad Y_i = \begin{cases} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{cases}$

$X_n =$ position of particle at time n $= \sum_{i=1}^{n} Y_i \qquad X_0 = 0$

$$|X_k - X_{k-1}| \le 1 \implies \sum_{k=1}^{t} c_k^2 = t$$

$\Pr(|X_t - X_0| > \lambda) \le 2e^{-\frac{\lambda^2}{2t}}$ by Azuma-Hoeffding

$t = n \quad \lambda = n^{1/2+\varepsilon}$

$\Pr($particle $\gg \sqrt{n}$ steps from origin after n steps$)$

$$\le 2e^{-\frac{n^{1+2\varepsilon}}{2n}} = O(e^{-n^{\varepsilon}})$$

Chromatic # in random graph $G(n, \frac{1}{2})$

Azuma $\implies$ sharp concentration of chromatic # around its expectation vertex exposure martingale $|X_k - X_{k-1}| \leq 1$

Finding "interesting" patterns (e.g. in DNA sequences)

Let $X = (X_1, .., X_n)$ be sequence of characters chosen independently and u.a.r. from Σ $|\Sigma| = s$ (e.g. $\Sigma = \{A, T, C, G\}$)

$B = (b_1, ..., b_k)$ fixed string of characters.

F : # occurrences of F

$E(F) = ?$

$$E(F) = (n-k+1)\left(\frac{1}{s}\right)^k$$

Let $Z_i = E(F | X_1, .., X_i)$ Doob martingale

$Z_n = F$, $Z_0 = E(F)$

$|Z_{i+1} - Z_i| \leq k$ since each character can participate in $\leq k$ matches

$\Rightarrow$ By Azuma-Hoeffding

$$\Pr(|F - E(F)| \geq \lambda) \leq 2e^{-\frac{\lambda^2}{2nk^2}}$$

or for $\lambda = ck\sqrt{n}$

$$\Pr(|F - E(F)| \geq ck\sqrt{n}) \leq 2e^{-c^2/2}$$