n people brought into room

again each person gets hat (Red or Blue) placed on their head

they can see other people's hats but not their own.

Without communicating, they each write down R or B.

If any of them are wrong, they will all be shot.

They can agree ahead of time on a protocol.

With what probability can they avoid being shot?

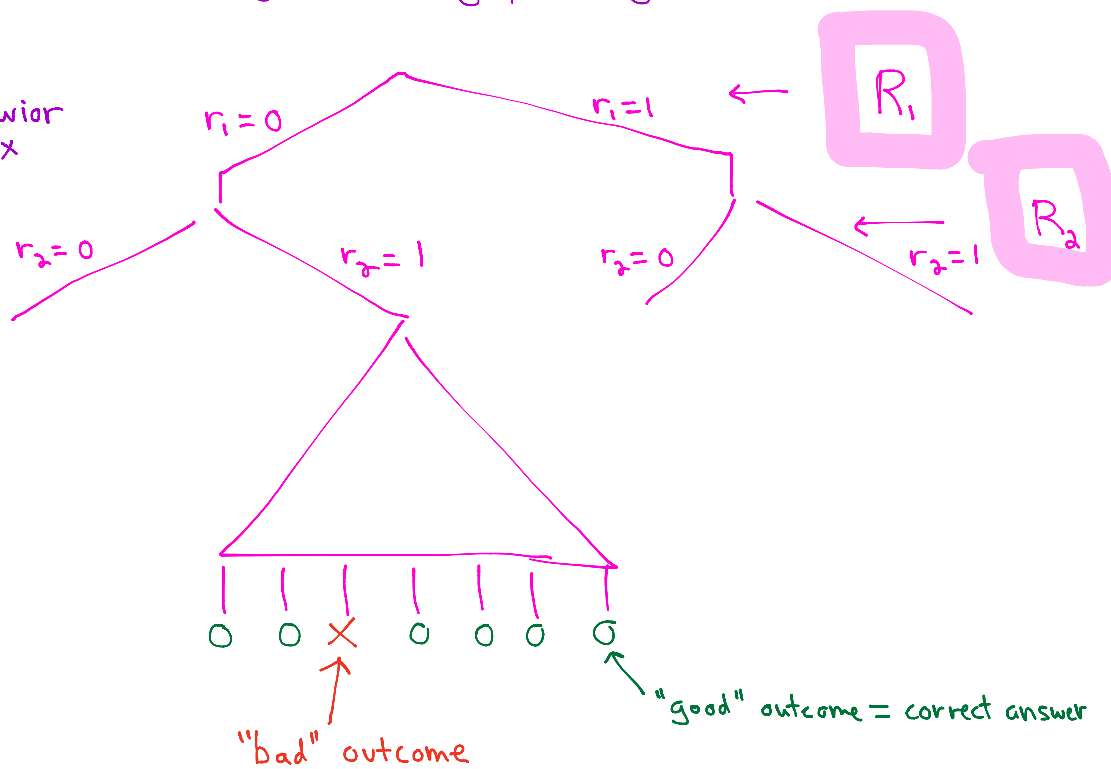
Method of Conditional Expectation

Consider randomized alg A that uses m random bits. $\Pr(A(x;R_1,\dots,R_m)$ good$)$ at least, say, $\frac{2}{3}$

sequences of coin tosses $\Leftrightarrow$ binary tree

"Good" randomized alg $\Rightarrow$ many paths good

Fix behavior on input x



$R_1,\dots,R_m$ seq of unif, indep random bits

Define $P(r_1,\dots,r_i)$ = fraction of continuations that are good.

$$= \Pr(A(x;R_1,\dots,R_m) \text{ good} \mid R_1=r_1,\dots R_i=r_i)$$

$$= E[P(r_1,\dots,r_i,R_{i+1})]$$

$$= \tfrac{1}{2}P(r_1,\dots,r_i,0) + \tfrac{1}{2}P(r_1,\dots,r_i,1)$$

$\Rightarrow \exists\, r_{i+1} \in \{0,1\}$ s.t. $P(r_1,\dots,r_{i+1}) \geq P(r_1,\dots,r_i)$

To find good path, just walk down tree & pick

$r_i \in \{0,1\}$ for $i=1..m$ s.t. $P(r_1,\dots,r_{i+1}) \geq P(r_1,\dots,r_i)$

At end:

$$P(r_1,\dots,r_m) \geq P(r_1,\dots,r_{m-1}) \geq \dots \geq P(r_1) \geq P(A(x;R_1,\dots,R_m)) \geq \frac{2}{3}$$

0 or 1 $\Rightarrow$ must be 1

Issue: to do this need to be able to deterministically

compute $P(r_1,\dots,r_i)$; may be infeasible

but sometimes works

Reconsider randomized alg for MAX CUT

$G=(V,E)$ $\quad V=\{v_1,\dots,v_n\}$ $\quad E=\{e_1,\dots,e_m\}$

for $i := 1$ to m

Let $R_i \in_{\text{random}} \{0,1\}$

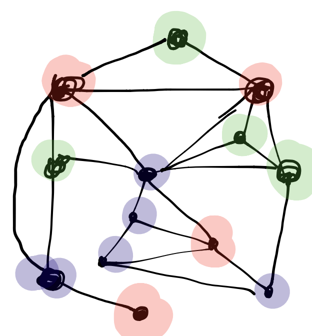
If $R_i = 0$, put v_i in S, else put $v_i \in \bar{S}$

Derandomization using method of conditional expectations

$$C(r_1,\dots,r_i) \triangleq E(\text{cut}(S,\bar{S}) \mid R_1 = r_1, \dots, R_i = r_i)$$

$$C(\phi) = E(\text{cut}(S,\bar{S})) = \frac{|E|}{2}$$

Need to compute $C(r_1,\dots,r_i)$



$S_i = \{v_j \mid j \le i, R_j = 0\}$

$\bar{S}_i = \{v_j \mid j \le i, R_j = 1$

U_i undecided $= \{v_{i+1}, \dots, v_n\}$

$$C(r_1,\dots,r_i) = |\text{cut}(S_i,\bar{S}_i)| + \frac{1}{2}|\#\text{edges with one endpoint in } U_i|$$

$\Rightarrow$ can deterministically select value for r_{i+1} by comparing

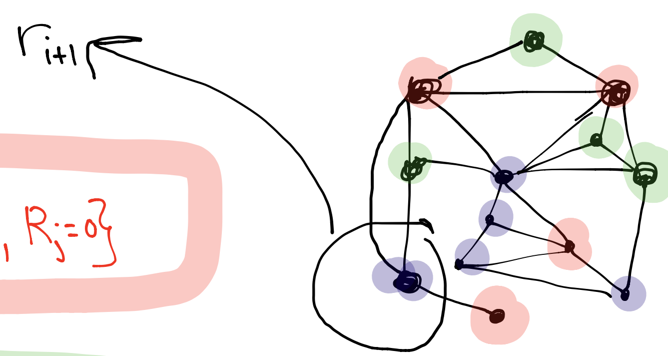
$C(r_1,\dots,r_i,0)$ with $C(r_1,\dots,r_i,1)$

Observation:

$$C(r_1,\dots,r_{i+1}) = |cut(S_{i+1},\bar{S}_{i+1})| + \frac{1}{2}\,\underbrace{|\#\text{edges with one endpoint in } U_{i+1}|}_{\text{independent of } r_{i+1}}$$

$\Rightarrow$ suffices to set r_{i+1} to maximize

$$|cut(S_{i+1},\bar{S}_{i+1})|$$



$S_i = \{v_j \mid j \le i,\ R_j = 0\}$

$\bar{S}_i = \{v_j \mid j \le i,\ R_j = 1$

U_i undecided $= \{v_{i+1}, \dots, v_n\}$

$$|cut(S_{i+1},\bar{S}_{i+1})| = |cut(S_i,\bar{S}_i)| + \begin{cases} \#\text{edges from } v_{i+1} \text{ to } S_i & \text{if } r_{i+1} = 1 \\ \#\text{edges from } v_{i+1} \text{ to } \bar{S}_i & \text{if } r_{i+1} = 0 \end{cases}$$

To maximize, pick bigger one.

The greedy algorithm!

Corollary: The greedy algorithm is guaranteed to find a cut of size $\geq \frac{|E|}{2}$

[followed presentation in Vadhan Pseudorandomness]

Min Cut [MU] 1.4

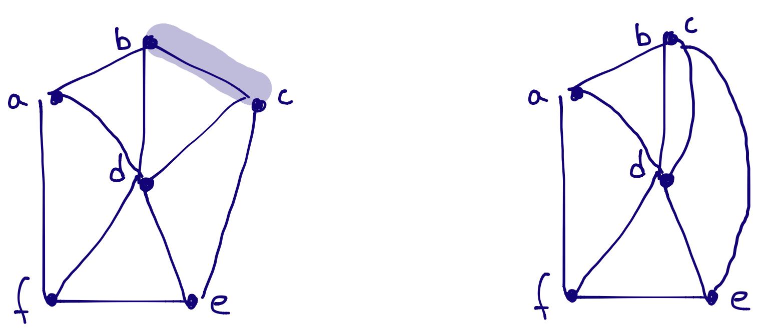
Given undirected multigraph, find minimal set of edges that disconnects graph

How to solve?

- find all s-t min cuts $\tilde{O}(mn \cdot n)$
- better alg $\tilde{O}(mn)$ [Hao, Orlin]

Key idea: contract edges

suppose could identify edge that didn't belong to min cut then can contract that edge to get smaller graph with same min cut.



$G \longrightarrow G/(b,c)$

Observation: random edge unlikely to be in min cut.

Repeat until G has 2 vertices

choose an edge (u,v) u.a.r.

Let $G := G/(u,v)$

return unique cut defined by contracted graph

can be implemented in $O(n^2)$ time

Lemma

A particular min-cut returned with prob $\geq \frac{1}{\binom{n}{2}}$

Pf Consider a particular min cut with c edges in it.

$\Pr(\text{select edge in this cut}) = \frac{c}{m}$ Claim $\leq \frac{2}{n}$

since mincut $= c \Rightarrow$ min deg $= c$

$\Rightarrow \#\text{edges} \geq \frac{nc}{2}$

when r vertices left

min cut $\geq c \Rightarrow$ mindeg $\geq c$

$\Rightarrow$ # edges $\geq \frac{rc}{2}$

$$\Pr(\text{choose one of these}) \leq \frac{c}{\frac{rc}{2}} \leq \frac{2}{r}$$

$$\Pr(\text{finding min cut}) = \Pr(\text{never contract min cut edge})$$

$$= \Pr(\text{don't on first}) \cdot \Pr(\text{don't on } 2^{nd} \mid \text{don't on } 1^{st})$$

$$\cdot \Pr(\text{don't on } 3^{rd} \mid \text{don't on } 1^{st} \text{ or } 2^{nd}) \cdots \Pr(\text{don't on last} \mid \text{don't till then})$$

$$\leq \left(1-\frac{2}{n}\right)\left(1-\frac{2}{n-1}\right)\cdots\left(1-\frac{2}{3}\right)$$

$$= \left(\frac{n-2}{n}\right)\left(\frac{n-3}{n-1}\right)\left(\frac{n-4}{n-2}\right)\cdots\left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{1}{3}\right)$$

$$= \left(\frac{\cancel{n-2}}{n}\right)\left(\frac{\cancel{n-3}}{n-1}\right)\left(\frac{\cancel{n-4}}{\cancel{n-2}}\right)\cdots\left(\frac{\cancel{3}}{\cancel{5}}\right)\left(\frac{2}{\cancel{4}}\right)\left(\frac{1}{\cancel{3}}\right)$$

$$= \frac{2}{n(n-1)} = \frac{1}{\binom{n}{2}} \geq \frac{1}{n^2}$$

To boost prob of success:

Repeat $n^2 \ln n$ times

$\Pr(\text{none of iterations succeeds}) \leq \left(1 - \frac{1}{n^2}\right)^{n^2 \ln n}$

Most useful approx ever!

$(1-x) \leq e^{-x}$

$\Rightarrow \quad (1-x)^y \leq e^{-xy}$

Taylor series:

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$\Rightarrow e^x \geq 1 + x \quad \forall x \in \mathbb{R}$

Another way to see this:

e^x convex fn $\Rightarrow$

linear approx at any pt lies beneath fn

$f(y) \geq f(x) + f'(x)(y-x)$

apply this at $x=0$

$\Pr(\text{none of iterations succeeds}) \leq \left(1 - \frac{1}{n^2}\right)^{n^2 \ln n} \leq e^{-\ln n} = \frac{1}{n}$

Running time: $O(n^4 \ln n)$

not great as min-cut alg

but already amazing consequences

How can we improve this?

Observation:

failure prob $\uparrow$ on later iterations, yet we're repeating everything when there are few vertices

if contract $n \to k$

$\Pr(\text{contract edge in min cut})$

$$= \left(\frac{\cancel{n-2}}{n}\right)\left(\frac{\cancel{n-3}}{n-1}\right)\cdots\left(\frac{k+1}{\cancel{k+3}}\right)\left(\frac{k}{\cancel{k+2}}\right)$$

$$\approx \frac{k^2}{n^2} \quad \text{so if } k = \frac{n}{\sqrt{2}}$$

$$\Pr(\text{no failure } n \to k) \le \frac{1}{2}$$

Recursive-Contract(G, n)

If G has 2 vertices

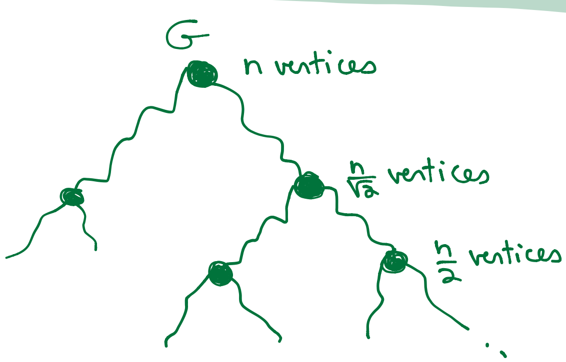
return weight of (unique) min cut

else repeat twice

$G' :=$ Contract G $\left(n \to \frac{n}{\sqrt{2}}\right)$ vertices

Recursive-Contract$\left(G', \frac{n}{\sqrt{2}}\right)$

return better of 2 results.



running time $T(n)$?

prob of success?

Running time:

$$T(n) = 2\left(n^2 + T\left(\frac{n}{\sqrt{2}}\right)\right)$$

$$T(n) \le cn^2 \log n$$

inductive step... roughly

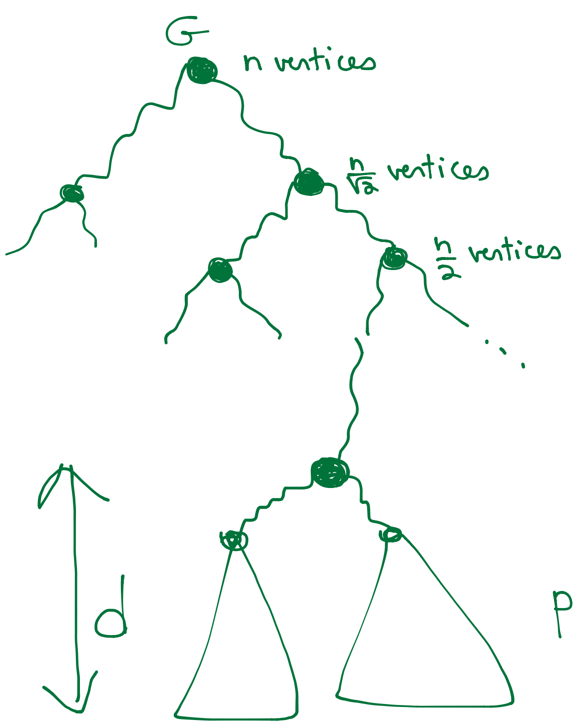
$$2n^2 + 2c\left(\frac{n}{\sqrt{2}}\right)^2 \log\left(\frac{n}{\sqrt{2}}\right)$$

$$cn^2 \log n - cn^2 \log\sqrt{2} + 2n^2$$

$$cn^2 \log n - \frac{cn^2}{2} + 2n^2 \le cn^2 \log n$$

$$\text{if } \frac{c}{2} \ge 2 \equiv c \ge 4$$

Probability of success



Define

$$p_d = \Pr(\text{success in subtree} \mid \text{no good edge contracted so far})$$

Recurrence for p_d:

$$p_d = 1 - \Pr(\text{both children fail})$$

$$= 1 - (1 - \underbrace{\tfrac{1}{2} p_{d-1}}_{\text{success}})^2$$

$$p_0 = 1$$

$$\boxed{\text{Claim: } p_d \geq \frac{1}{d+1}}$$

$$p_d = 1 - (1 - \tfrac{1}{2} p_{d-1})^2$$

$$= 1 - \left(1 - p_{d-1} + \frac{p_{d-1}^2}{4}\right)$$

$$= p_{d-1} - \frac{p_{d-1}^2}{4}$$

$$\frac{1}{d} - \frac{1}{4d^2} \geq \frac{1}{d} - \frac{1}{d(d+1)} = \frac{1}{d}\left(\frac{d+1-1}{d+1}\right) = \frac{1}{d+1}$$

$$\frac{1}{4d^2} \leq \frac{1}{d(d+1)} \quad \forall d \geq 1$$

What is max d?

depth i #vertices $\frac{n}{2^{i/2}}$

$$2^{d/2} = n \Rightarrow \log n = \frac{d}{2}$$

$$\max d = 2 \log n$$

$\Rightarrow \Pr(\text{success}) = \Omega\left(\frac{1}{\log n}\right)$

to get failure prob ε run k times

$$\left(1 - \frac{1}{\ln n}\right)^k \le \varepsilon \qquad (1-x) \le e^{-x}$$

$$e^{-\frac{k}{\ln n}} \le \varepsilon \qquad (1-x)^y \le e^{-xy}$$

$$\frac{k}{\ln n} \ge \ln\left(\frac{1}{\varepsilon}\right)$$

$k \geq \ln\left(\frac{1}{\varepsilon}\right) \ln n$ For example $\varepsilon = \frac{1}{n}$

$k = (\ln n)^2$

$\boxed{\text{Theorem [Karger, Karger/Stein]}}$

There is a randomized Monte Carlo algorithm for min-cut that runs in time $\tilde{O}(n^2)$. It fails to output a min cut with probability $O\left(\frac{1}{n^k}\right)$.

Random choices affect both running time & correctness

$\boxed{\text{Monte Carlo algorithm:}}$ has fixed (deterministic) running time but has small prob of giving wrong answer.

$\boxed{\text{Las Vegas algorithm:}}$ always gives correct answer but running time is r.v.

If it runs in time $f(n)$ w.h.p. then can convert to Monte Carlo by just stopping after time $f(n)$

No universal method known for Monte Carlo → Las Vegas

Problem: sometimes don't know how (or impossible) to check correctness

$\boxed{\text{Neat Corollary}}$

How many min cuts can there be?

$\boxed{\text{Lemma}}$ In any graph, there are at most $\binom{n}{2}$ min cuts

Proof In basic contraction algorithm, prob a particular min cut survives $\geq \frac{1}{\binom{n}{2}}$

Let $C_1, \ldots, C_r$ be enumeration of min cuts

$$\Pr(\exists i \text{ s.t. } C_i \text{ survives contraction})$$

$$= \sum_{i=1}^{r} \Pr(C_i \text{ survives})$$

only 1 cut survives

survival of different cuts are disjoint events

$$1 \geq \Pr(\exists i \text{ s.t. } C_i \text{ survives contraction})$$

$$= \sum_{i=1}^{r} \Pr(C_i \text{ survives})$$

$$\geq r \frac{1}{\binom{n}{2}}$$

$$\Rightarrow \# \underset{(r)}{\text{min cuts}} \leq \binom{n}{2}$$

tight for cycle.

Similar result for approximate min cuts

Corollary:

$\forall$ undirected graph, $\alpha \geq 1$, # of $\underbrace{\alpha\text{-min cuts}}_{\text{\# of edges} \leq \alpha \text{ size of min cut}} \leq n^{2\alpha}$

Pf: assume α half integral

Let $r = 2\alpha$

Run contraction alg until only r vertices

$\Rightarrow$ # cuts 2^r

(if stop contraction early α-min cut has reasonable chance of surviving)

Output one of these u.a.r.

Pr(a particular α-min cut survives contraction to r)?

when k vertices left, # edges $\geq \frac{kc}{2}$

Prob pick edge in particular α-mincut $\leq \frac{\alpha c}{kc/2} = \frac{2\alpha}{k}$

Pr(a particular α-min cut survives contraction to r)

$$\geq \left(1-\frac{2\alpha}{n}\right)\left(1-\frac{2\alpha}{n-1}\right)\cdots\left(1-\frac{2\alpha}{r+1}\right)$$

$$= \left(\frac{n-2\alpha}{n}\right)\left(\frac{n-1-2\alpha}{n-1}\right)\cdots\left(\frac{r+1-2\alpha}{r+1}\right)$$

$$= \frac{(n-2\alpha)!}{(r-2\alpha)!}\,\frac{r!}{n!}$$

$$\Rightarrow \Pr(\text{particular } \alpha\text{-min cut } \boxed{\text{output}}) \geq \frac{1}{2^r}\,\frac{(n-2\alpha)!}{(r-2\alpha)!}\,\frac{r!}{n!}$$

$$\geq \frac{1}{\cancel{r!}}\cdot\frac{(n-2\alpha)!}{(r-2\alpha)!}\,\frac{\cancel{r!}}{n!} \quad (r \geq 4)$$

$$\geq \frac{1}{n^{2\alpha}}\,\frac{1}{(r-2\alpha)!} \geq n^{-2\alpha}$$

$C_1, \ldots, C_t$ set of α-min cuts

$$1 \geq \Pr(\text{some } C_i \text{ output by alg})$$

$$= \sum_{i=1}^{t} \Pr(C_i \text{ output})$$

$$\geq t\, n^{-2\alpha}$$

$$\Rightarrow t \leq n^{2\alpha}$$