$P_{xy} = \dfrac{c_{xy}}{c_x} \qquad \pi_x = \dfrac{c_x}{c_G} \qquad G$ connected

$$L(G) = \begin{pmatrix} c_1 & & & 0 \\ & c_2 & & \\ & & \ddots & \\ 0 & & & c_n \end{pmatrix} - \begin{array}{c c} & j \\ i & \begin{pmatrix} & & & \\ & & -c_{ij} & \\ & & & \\ & & & \end{pmatrix} \end{array}$$

$\boxed{\text{Laplacian of } G}$

$$= \sum_e c_e \begin{array}{c c} u & \\ & \\ v & \end{array}\!\!\begin{pmatrix} 1 \\ \vdots \\ -1 \end{pmatrix} \begin{array}{c} \begin{array}{c c} u & v \end{array} \\ \begin{pmatrix} 1 & \cdots & -1 \end{pmatrix} \end{array} \qquad e = (u, v)$$

$$x^T L(G) x = \sum_{e=(u,v)} c_e (x_u - x_v)^2$$

Laplacian p.s.d. $\Rightarrow \lambda_i \geq 0$

$0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq$

all 1's eigenvector

If G connected $\lambda_2 > 0$

Laplacian quadratic form highlights connection between Laplacian and cut structure

take $\vec{x}$ so $x_i = \begin{cases} 1 & i \in S \\ 0 & i \notin S \end{cases}$

then $x^T L_G x = \sum_{\substack{e=(u,v) \\ u \in S,\, v \in \bar{S}}} c_e$ measures capacity of cut

no small cuts $\equiv$ "expansion" $\equiv$ rapid mixing $\equiv$ spectral gap

$\hat{\lambda}_2$ 2^{nd} smallest eigenvalue of normalized Laplacian

$$\hat{L}(G) = D(G)^{-1/2} L(G) D(G)^{-1/2}$$

$T(\varepsilon)$ # steps for lazy r.w. to converge to within ε of π

$$\left| q^0 P^{T(\varepsilon)}_{(j)} - \pi_j \right| \leq \varepsilon \quad \forall j$$

$$T(\varepsilon) = \frac{1}{\hat{\lambda}_2} \log\left(\frac{n}{C\varepsilon}\right) \qquad C = \frac{\max_e c_e}{\min_e c_e}$$

Connections to cut structure

$$\Phi_G = \underset{\text{of graph}}{\text{conductance}} = \min_{\emptyset \neq S \subset V} \phi(S)$$

$$\phi(S) = \frac{\sum_{\substack{e \text{ crosses} \\ S-\bar{S}}} c_e}{\min\left(\sum_{\substack{e \text{ incident} \\ \text{to } v \text{ in } S}} c_e , \sum_{\substack{e \text{ incident} \\ \text{to } v \text{ in } \bar{S}}} c_e \right)}$$

Cheeger's Inequality:

$$2\Phi_G \geq \hat{\lambda}_2 \geq \frac{\Phi_G^2}{2}$$

cuts w/ small conductance are obstacles to rapid mixing

in some sense only obstacles

Back to electrical flows:

$\forall$ electric flow $\vec{f}$, $\exists$ vector ϕ of vertex potentials

for unit s-t current, compute $\vec{\phi}$ by solving system

$$L_G \phi = \begin{array}{c} \begin{pmatrix} 1 \\ \\ \\ -1 \end{pmatrix} \end{array} \begin{array}{l} s \\ \\ \\ t \end{array}$$

$$c_x \phi_x - \sum_{y \in N(x)} c_{xy} \phi_y = 0$$

$$c_s \phi_s - \sum_u c_{su} \phi_u = 1$$

$$\sum_u c_{su}(\phi_s - \phi_u) = \sum_u f_{su} = 1$$

$$\mathcal{E}_r(f) = f^T R f = \phi^T L(G) \phi$$

diagonal $m \times m$ matrix of edge resistances

$$\| $$

$$\sum_e c_e(\phi_u - \phi_v)^2 = \sum_e c_e \frac{f_e^2}{c_e^2} = \sum_e r_e f_e^2$$

Applications

Spectral Sparsification

[Spielman/Teng] [Spielman/Srivastava]
.....

Given $G=(V,E,c)$ find $H=(V,F,w)$

sparse $|F|=O(n)$

s.t. H approximates G well

$\equiv$ all cuts approximately preserved ++

$$x^T L_H x \approx x^T L_G x \qquad \forall \vec{x} \qquad (\text{eigenvalues} \approx \text{same})$$

could be const here

Example: degree d expanders approx complete graph

sample edges w/ prob $\propto$ effective resistance between endpts

Solving symmetric diagonally dominant linear systems

$Ax = b$ A symm $A_{ii} \geq \sum_{j \neq i} |A_{ij}|$

[Spielman Tang] . . . . [Kelner, Orecchia, Sidford, Zhu]

Numerous applications: scientific computing, machine learning, vision, ...
+ algorithmic applications – flow, sparsification, routing, etc.

reduces $\longrightarrow$ $L\phi = \chi_{s,t}$ $\equiv$ electrical flow problem

New alg: start w/ unit s-t flow (e.g. s-t path)

If it was electrical flow $\forall$ cycle C $\sum_{e \in C} r_e f_{\vec{e}} = 0$

– randomly sample cycle from well-chosen dist'n

– compute $\sum_{e \in C} r_e f_e$ & if not 0, add flow around cycle

to make this 0.

show that w/ suitable data structures can be

implemented efficiently in not too many cycle updates

Max Flow

[Christiano et al, Srivastava/Rao]

- compute electrical flow by solving $L\phi = \chi_{s,t}$ efficiently
  capacities will be violated

Version 1 penalize congested edges by $\uparrow r_e$
repeat

Version 2 $(u \to v)$ overly congested
$\Rightarrow$ create current sce at v of value = excess flow
remove current at u
repeat

Random Spanning Trees

Procedure: Fix ordering on vertices $v_1 = r, \ldots, v_n$

root

Take rw starting at v_2 till hit r, erasing any cycles.

Add directed edges on path to tree

If T spanning stop

If not, take first vertex in ordering that isn't visited

Take r.w erasing cycles until hit tree

repeat.... till tree is spanning

This tree is uniformly random tree!!! [Wilson]

Claim: Suppose fix a root r & select oriented tree

w/ edges directed towards root with prob $\propto \prod_{e \in T} p_{\vec{e}}$

Then if orientation & root are forgotten

$$\Pr(T) \propto \prod_{e \in T} c_e$$

fix r: $\Pr(\vec{T}^r) \propto \prod_{e \in T} p_{\vec{e}} = \dfrac{\prod_{e \in T} c_e}{\prod_{v \in V \setminus r} c_v}$

$\propto \dfrac{\prod_{e \in T} c_e}{\prod_v c_v}$ since c_r same for all $\vec{T}^r$

$\propto \prod_{e \in T} c_e$ since denom doesn't depend on T

Theorem

Wilsons alg finds tree in $\vec{T}^r$ w/ prob $\propto \prod p_{\vec{e}}$

therefore once root & orientations forgotten, find tree

w/ prob $\propto \prod_{e \in T} c_e$

Proof: Order in which cycles popped off stack doesn't matter

Lemma

$$\Pr((x,y)\in T) = c_{(x,y)} R_{x\leftrightarrow y}$$

Proof

Run Wilson's alg with root y, starting at x

$$\Pr(e\in T) = \Pr_x(\text{first hit } y \text{ walking along } e)$$

$$= \sum_t \Pr(\text{return to } x \text{ at time } t \text{ w/ seeing } y)\frac{c_e}{c_x}$$

$$= E_x(\#\text{visits to } x \text{ w/ seeing } y)\frac{c_e}{c_x}$$

$$= c_x R_{x\leftrightarrow y}\frac{c_e}{c_x} = c_e R_{x\leftrightarrow y}$$

Negative correlation: $\Pr(e\in T \text{ and } f\in T) \leq \Pr(e\in T)\Pr(f\in T)$

Proof:

$$\Pr(e\in T_G \cap f\in T_G) = \Pr(e\in T_G)\Pr(f\in T_G \mid e\in T_G)$$

$$= \Pr(e\in T_G)\Pr(f\in T_{\underbrace{G/e}_{\text{endpts of } e \text{ fused}}})$$

$\equiv$ resistance 0

$\leq \Pr(f \in T_G)$

$\uparrow$

Rayleigh Monotoncity

These results used to get best known alg for TSP problems

Kirchoff's Matrix Tree Thm

$$\sum_{T} \prod_{e \in T} c_e = \frac{1}{n} \prod_{i=2}^{n} \lambda_i$$

eigenvalues of Laplacian