$\boxed{\text{general random walk on graph}}$ (reversible)

Associated with each edge $e=(x,y)$ is

$C_{xy} = C_e$ "conductance of edge"

$$\boxed{P_{xy} = \frac{C_{xy}}{C_x}}$$

$$C_x = \sum_{y \mid (x,y)\in E} C_{xy}$$

simple random walk $\quad C_{xy}=1 \quad \forall (x,y)\in E$

$$\pi_x = \frac{C_x}{C_G} \qquad C_G = \sum_{x\in V} C_x$$

$\boxed{\text{Connections with Electrical Networks}}$

a fn $h: V \to \mathbb{R}$ is **harmonic** for P at v if

$$h(i) = \sum_j P_{ij} h(j)$$

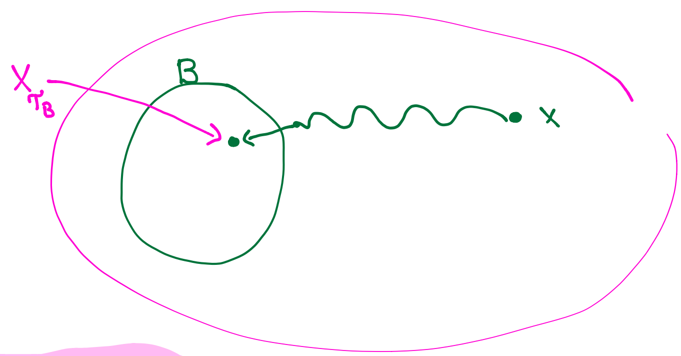
h is avg of neighbors

Proposition: (X_t) irreducible MC

$B \subset S$ $\quad h_B : B \to \mathbb{R}$ fn defined on B

$$h(x) = E_x h_B(X_{\tau_B})$$

τ_B : time to reach B



is unique harmonic extension of h_B

① $h(x) = h_B(x) \quad \forall x \in B$

② h harmonic on $x \notin B$

Proof

① ✓

(2) $h(x) = E_x h_B(X_{T_B}) = \sum_y P_{xy} \underbrace{E_x[h_B(X_{T_B}) | X_1 = y]}_{E_y[h_B(X_{T_B})]}$

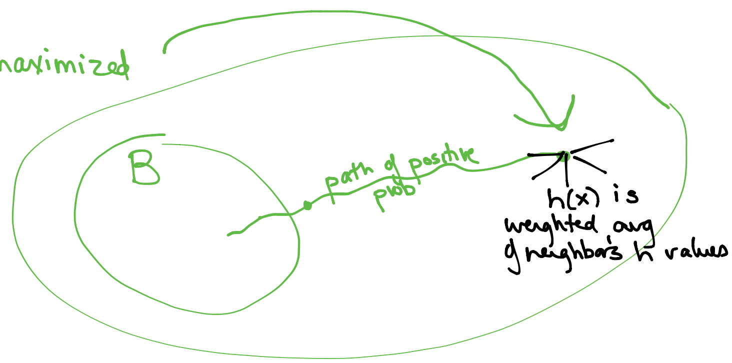
$= \sum_y P_{xy} h(y)$ ✓

Uniqueness: Suppose not. h & g both harmonic extensions

then $h(x) - g(x)$ $= 0$ on $x \in B$

is harmonic on $x \notin B$

Consider x where $h(x)-g(x)$ maximized



$\Rightarrow$ max ≤ 0

similarly min ≥ 0

$\Rightarrow h(x) = g(x) \quad \forall x$

Electrical Networks, Voltages & Current Flows

$G = (V, E)$

Fix 2 nodes s & t

source sink

A fn ϕ on V which is harmonic on $V \setminus \{s,t\}$

called a **voltage**

(completely determined by $\phi(s)$ & $\phi(t)$)

$e = (x,y)$

An (electrical) flow is a fn on <u>oriented</u> edges $\vec{e} = x \to y$, $\overleftarrow{e} = x \leftarrow y$

is

- anti-symmetric $f(\vec{e}) = -f(\overleftarrow{e}) \quad \forall e$
- $\sum_{y \mid (x,y) \in E} f(x \to y) = 0 \quad \forall x \notin \{s,t\}$
- $\sum_{y \mid (s,y) \in E} f(s \to y) \geq 0$

flow out of $s \geq 0$

Kirchhoff's Law = flow in = flow out

$$\sum_x \sum_{y \mid (x,y) \in E} f(x \to y) = 0$$

strength of flow
or value

To construct electrical network $\forall$ edge e, put resistor

where $r_e = \dfrac{1}{c_e}$.

resistance

conductance

Given voltage ϕ, the current flow I on each edge $\vec{e} = x \to y$ satisfies

$\boxed{\text{Ohm's Law}}$

$$I(x \to y) = \frac{\phi(x) - \phi(y)}{r_{xy}} = c_{xy}\left[\phi(x) - \phi(y)\right]$$

resistance on edge

$\boxed{\text{Observations:}}$

(1) $x \notin \{s,t\}$

$$\sum_{y \mid (x,y) \in E} I(x \to y) = \sum_{y \mid (x,y) \in E} \phi(x) - \phi(y) = d_x \phi(x) - \sum_{y \mid (x,y) \in E} \phi(y)$$

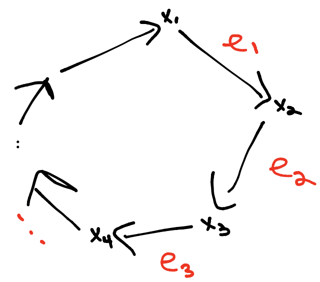
$$= d_x \underbrace{\left[\phi(x) - \sum_{y \mid (x,y) \in E} \frac{1}{d_x} \phi(y)\right]}_{=0 \text{ since } \phi \text{ harmonic}}$$

$\boxed{\text{Conservation of current}}$

$$\sum_{y \mid (x,y) \in E} I(x \to y) = \sum_{y \mid (x,y) \in E} c_{xy}\left(\phi(x) - \phi(y)\right) = \dots = 0$$

② Cycle law

$$\sum_i r_{e_i} I(x_i \to x_{i+1}) = \sum_i \phi(x_i) - \phi(x_{i+1})$$
$$= 0$$



③ $\phi'(x) = \phi(x) + c \quad \forall x$ has no effect on current or harmonicity

$\Rightarrow$ wlog $\phi(t) = 0$ & ϕ determined by $\phi(s)$

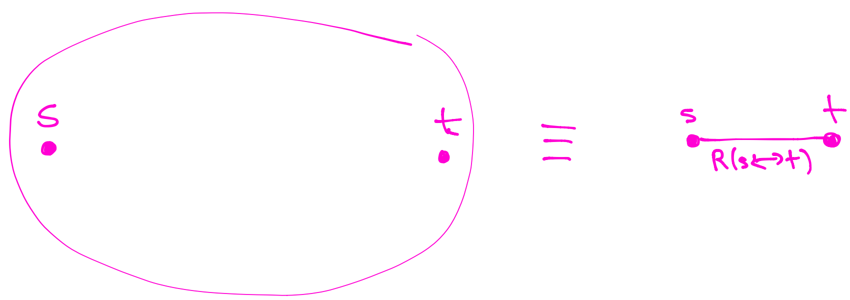
If f is a flow satisfying the cycle law for every cycle, and value of flow = value of current flow, then $f = I$

given resistances on edges

ϕ voltage $\iff$ I current flow

Definition The **effective resistance** between s & t

$$R(s\leftrightarrow t) = \frac{\phi(s)-\phi(t)}{\|I\|} \quad \leftarrow \text{indep of } \phi$$



$\boxed{\text{Lemma 1}}$

$$\Pr\left(T_{st} < T_{ss}^{+}\right) = \frac{1}{d_s R(s \leftrightarrow t)}$$

$\boxed{\text{Proof}}$

$$g : x \longmapsto \Pr_x\left(T_{xt} < T_{xs}\right)$$

is a harmonic fn on $V \setminus \{s,t\}$

s.t. $g(s) = 0$ $\quad g(t) = 1$

$$h(x) = \frac{\phi(s) - \phi(x)}{\phi(s) - \phi(t)}$$

also harmonic with same boundary values

$$\Longrightarrow \quad \Pr_x\left(T_{xt} < T_{xs}\right) = \frac{\phi(s) - \phi(x)}{\phi(s) - \phi(t)}$$

$$\Pr_s\left(T_{st} < T_{ss}^{+}\right) = \sum_x P_{sx} \Pr_x\left(T_{xt} < T_{xs}\right)$$

$$= \sum_x \frac{c_{sx}}{c_s} \frac{\phi(s) - \phi(x)}{\phi(s) - \phi(t)}$$

$$= \frac{\sum_x I(s \to x)}{c_s(\phi(s) - \phi(t))} = \frac{\|I\|}{c_s(\phi(s) - \phi(t))}$$

$$= \frac{1}{c_s R(s \leftrightarrow t)}$$

$\boxed{\text{Lemma 2}}$

$$E_s(\#\text{ of visits to } s \text{ before hit } t) = c_s R(s \leftrightarrow t)$$

Proof: Lemma 1 $\Rightarrow$ geometric dist'n

$\boxed{\text{Lemma 3}}$

Let T be a $\boxed{\text{stopping time}}$ in finite irreducible MC s.t. $\Pr_s(X_T = s) = 1$

Then
$$\frac{E_s(\#\text{ of visits to } x \text{ before } T)}{E_s(T)} = \pi_x \qquad \forall x$$

Commute Time Identity

Consider r.w. on G (c_e conductance on edge e)

$$P_{xy} = \frac{c_{xy}}{\sum_{y \mid (x,y)\in E} c_{xy}} = \frac{c_{xy}}{c_x}$$

$\forall s,t$

$$\boxed{E(T_{st}) + E(T_{ts}) = c_G R(s \leftrightarrow t)} \qquad c_G = \sum_{x\in V} c_x$$

$\boxed{\text{Pf}}$ Use Lemma 3

Let $\tau = T_{st} + T_{ts}$

$$\frac{E_s(\#\text{ of visits to } s \text{ before } \tau)}{E_s(\tau)} = \pi_s = \frac{c_s}{c_G}$$

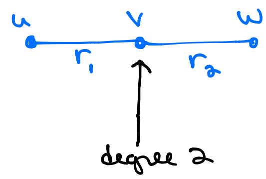
$$= E_s(\#\text{ of visits to } s \text{ before reaching } t)$$

$$\overset{\text{Lemma 2}}{=} c_s R(s \leftrightarrow t)$$

Computing effective resistance

sometimes can simplify network w/o changing quantities such as effective resistance

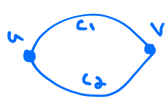
① Series Law: resistance in series add



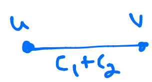
$\Longrightarrow$

voltages & currents unchanged

② Parallel Law: conductances in parallel add



$\Longrightarrow$



$$r = \frac{1}{\frac{1}{r_1}+\frac{1}{r_2}} = \frac{r_1 r_2}{r_1+r_2}$$

Some applications of commute time identity

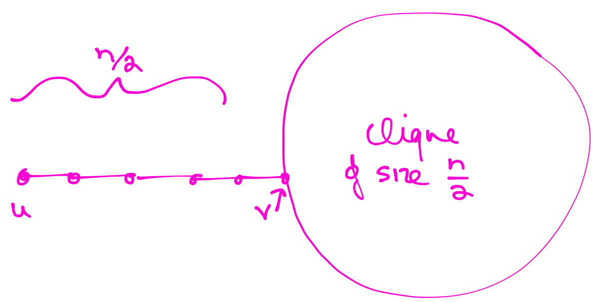
① Random walk on line

$R_{0 \leftrightarrow n} = n$ by series law

$c_G = n$

$\Rightarrow h_{0n} + h_{n0} = 2n^2$

② lollilop



$c_G = m = n^2$

$R_{u \leftrightarrow v} = \frac{n}{2}$

$\Rightarrow h_{uv} + h_{v0} = \Theta(n^3)$

Cover Time Bounds

Thm: $C(G) = \Omega(mR)$ where $R = \max_{u,v} R_{u \leftrightarrow v}$

$C(G) = O(mR \log n)$

Lower bound immediate.

for upper bound, consider $\ln n$ epochs of length $\underbrace{a}_{\text{const to be determined}} mR$

$\forall v \quad \forall u \qquad E_u(T_v) \le 2mR$

$\Rightarrow \Pr(v \text{ not visited in particular epoch}) \le \frac{1}{a}$ by Markov Inequality

$\Rightarrow \Pr(v \text{ not visited in any of } \ln n \text{ epochs}) \le \left(\frac{1}{a}\right)^{\ln n} = n^{-\ln(a)}$

$\Rightarrow \Pr(\exists \text{ vertex not visited}) \underset{\text{union bound}}{\le} n^{1-\ln a}$

$$C(G) \le 2amR\ln n \quad + n^{1-\ln a} \underbrace{n^3}_{\text{worst case bound}}$$

with $a=e^4$ $\qquad$ $C(G) = O(mR\ln n)$

Example: complete graph

R ?

$H_{uv} = n-1$ $\quad$ $\left(\Pr(\text{hit } v \text{ each step}\right) = \frac{1}{n-1}$

$$H_{uv} + H_{vu} = 2mR_{u\leftrightarrow v}$$

$$\frac{2}{n-1} = 2mR_{u\leftrightarrow v} \qquad \Rightarrow R_{u\leftrightarrow v} = \frac{2}{n}$$

$$C(G) = O\left(mR\ln n\right) = O\left(\binom{n}{2}\frac{2}{n}\ln n\right)$$

$$= O(n\ln n)$$

$\mathcal{E}(f)$ = energy of flow $f = \sum_e f(e)^2 r_e$

Thomson's Principle

For any finite connected graph

$R_{s\leftrightarrow t} = \inf\{\mathcal{E}(f) \mid f \text{ is unit flow from } s \rightsquigarrow t\}$

$\downarrow$

Unique minimizer is unit current flow

Corollary $\{r_e\}_{e\in E}$ & $\{r'_e\}_{e\in E}$ sets of resistances on same G

with $r_e \le r'_e \ \forall e$

Then $R_{s\leftrightarrow t} \le R'_{s\leftrightarrow t}$

Proof $\inf_f \sum r_e f_e^2 \le \inf_f \sum_e r'_e f_e^2$

+ Thomson's principle

Corollary: Adding an edge doesn't $\uparrow$ eff resistance