Puzzle

n prisoners brought into room. An infinite pile of hats each independently & equally likely to be Red or Blue. Without communicating they must each write down the index of the first red hat on their head.

They are all shot unless they are all right

Show that there is a strategy they can agree on ahead of time so that probability they survive is $\Omega\left(\frac{1}{\log n}\right)$.

Bipartite Matching

Given bipartite graph (L, R, E),

a matching $M \subseteq E$ is a set of edges that share no common endpoints

Common goal: find matching of maximum size

Online Bipartite Matching

L is known ahead of time

vertices in R arrive one at a time

When $j \in R$ arrives

learn which vertices in L are neighbors of j

make an irrevocable decision as to which neighbor if any to match j to.

Killer ap:

Internet ad allocation

match page views/ad slots with advertiser demand

Greedy Alg : match arriving node to any neighbor

always obtains matching of size $\geq \frac{1}{2}$ OPT

Analysis: every time Greedy adds an edge to the matching, think of it as earning \$1

Place 50¢ on each endpoint.

Now consider any edge (i,j) matched by OPT (when j arrives)

if Greedy matches j then charge OPT's edge to 50¢ on j

if Greedy can't match j, then i is already matched

charge OPT's edge to 50¢ on i

$\Rightarrow \forall$ \$1 OPT earns, Greedy earns $\geq$ 50¢.

The competitive ratio of Greedy $\left[:= \max_I \frac{\text{size of Greedy matching on } I}{\text{size of OPT matching on } I}\right]$
$= \frac{1}{2}$

No deterministic alg can do better.

So we turn to randomization ...

But first, an easier problem: fractional version

When a node $j \in R$ arrives, can allocate it fractionally.

Claim: Let A be a randomized alg for integral matching.

$\exists$ deterministic fractional alg D s.t. $\forall$ instance I

$$\sum_{i,j} x^D_{ij}(I) = E\left(\sum_{i,j} X^A_{ij}(I)\right)$$

D "simulates A"

when j arrives, set $x^D_{ij} = \Pr(X^A_{ij} = 1)$

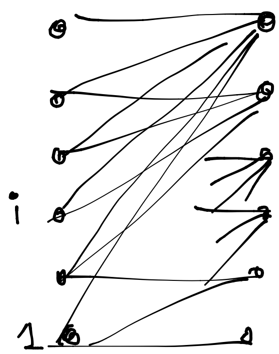
$$\forall i \sum_{j \in N(i)} X_{ij} \le 1 \implies E\left(\sum_{j \in N(i)} X_{ij}\right) \le 1 \implies \sum_{j \in N(i)} x_{ij} \le 1$$

$$\Rightarrow \sum_{(i,j) \in E} x_{ij} = E\left[\sum_{(i,j) \in E} X_{ij}\right] = E(\text{performance of } A)$$

$\Rightarrow$ upper bound on c.r. of deterministic fractional alg $\Rightarrow$

upper bound on c.r. of randomized alg.

Upper bound on fractional c.r.



i^{th} vertex gets $\frac{1}{n}+\frac{1}{n-1}+\dots+\frac{1}{i} = H_n - H_{i-1}$

$\approx \ln\frac{n}{i} \quad = 1$

when $i \approx \frac{n}{e}$

matching has size $n(1-\frac{1}{e})$

OPT matching has size n

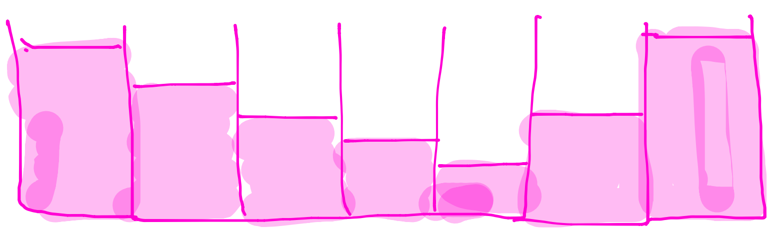
if don't divide equally worse

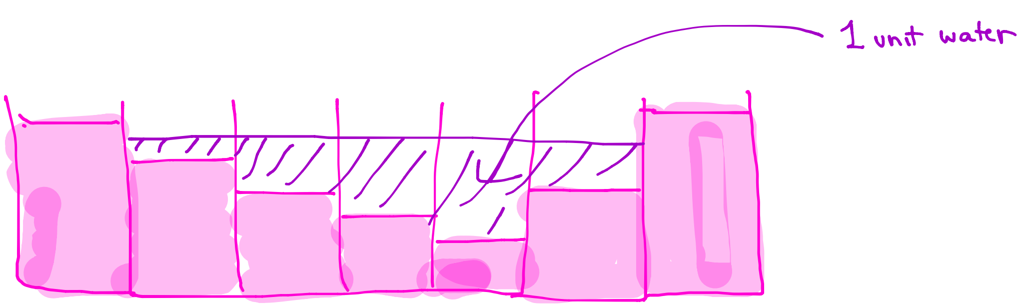
Water Level alg:

Imagine LHS water containers, capacity 1

RHS each vertex is a source of 1 unit of water

When vertex arrives, fill up neighbors in obvious way





Analysis:

again split gain between nodes, but when water level is high, want to leave relatively more money, to protect against possibility of mistake

When $x_{ij} \uparrow$ by dx increase α_i by $g(y_i)dx$

increase β_i by $(1-g(y_i))dx$

$g : [0,1] \to [0,1]$ $g(y)\uparrow$

$\boxed{\text{where } y_i := \sum_{j \in N(i)} x_{ij}}$

$$\sum_i \sum_j x_{ij} = \sum_i \alpha_i + \sum_j \beta_j$$

Want to choose $g(\cdot)$ to prove best possible c.r.

Proof outline: Suppose OPT assigns x^*_{ij} to edge (i,j)

Want to choose g so that no matter what:

$$(\alpha_i+\beta_j)\,x^*_{ij} \geq c\,x^*_{ij} \qquad (*)$$

for largest possible value of c

if so

$$\sum_i \alpha_i + \sum_j \beta_j \geq \sum_i \alpha_i \sum_{j\in N(i)} x^*_{ij} + \sum_j \beta_j \sum_{i\in N(j)} x^*_{ij}$$

$$= \sum_{(i,j)\in E} (\alpha_i+\beta_j)\,x^*_{ij} \underset{(*)}{\geq} \sum_{(i,j)\in E} c\,x^*_{ij} = c\,\mathrm{OPT}$$

Fix edge (i,j)

Let y_i^f be final water level in node i

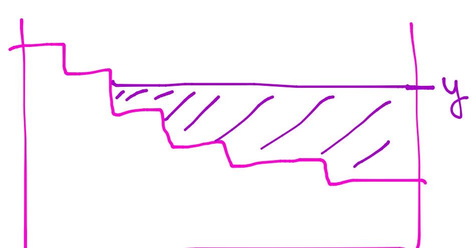
Case 1: $y_i^f = 1$ (x_{ij} could even be 0, in which case $\beta_j = 0$)

$$\Rightarrow \alpha_i = \int_0^1 g(x)\,dx = G(1)-G(0) \quad \text{where } G'(x)=g(x)$$

$\Rightarrow$ for c.r. c need $G(1)-G(0) = c$

Case 2: $y_i^f < 1 \Rightarrow j$ fully matched $\sum_{\tilde{i}\in N(j)} x_{\tilde{i}j} = 1$

Claim $\beta_j \geq 1-g(y) \geq 1-g(y_i)$



neighbors of j (i is one of them)

$$\alpha_i+\beta_j \geq G(y_i^f)-G(0) + 1-g(y_i^f)$$

want $G(y) - G(0) + 1 - g(y) \geq c \qquad \forall y \in [0,1]$

$$g(y) - g'(y) = 0 \implies g(y) = ke^{y}$$

Substituting back in $\displaystyle\int_0^y ke^x\,dx + 1 - ke^y = c$

$$\equiv ke^y - ke^0 + 1 - ke^y = 1 - k$$

to find k, use $G(1) - G(0) = c \qquad \equiv \displaystyle\int_0^1 ke^x\,dx = k(e-1)$

Combining we get $1 - k = k(e-1) \implies k = \frac{1}{e}$

$$\boxed{c = 1 - \frac{1}{e}}$$

optimal!

Equivalent to $\boxed{\text{primal-dual analysis}}$

Primal

$$\max \sum_{(i,j)\in E} x_{ij}$$

$$\sum_{j\in N(i)} x_{ij} \le 1 \quad \forall i \qquad \alpha_i$$

$$\sum_{i\in N(j)} x_{ij} \le 1 \quad \forall j \qquad \beta_j$$

$$x_{ij} \ge 0 \quad \forall (i,j)\in E$$

Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j$$

$$\alpha_i + \beta_j \ge 1 \quad \forall (i,j)\in E$$

$$\alpha_i \ge 0 \quad \forall i$$
$$\beta_j \ge 0 \quad \forall j$$

Weak duality $\Rightarrow$ feasible primal $\le$ OPT $\le$ feasible dual

We constructed near-feasible dual, by ensuring

that $\forall (i,j)\in E \quad \alpha_i + \beta_j \ge 1 - \frac{1}{e}$

$$\Rightarrow \hat{\alpha}_i = \frac{\alpha_i}{1-\frac{1}{e}} \quad \hat{\beta}_j = \frac{\beta_j}{1-\frac{1}{e}} \quad \text{feasible dual}$$

$$\Rightarrow \text{OPT} \le \sum_i \hat{\alpha}_i + \sum_j \hat{\beta}_j = \frac{1}{1-\frac{1}{e}}\left[\sum_i \alpha_i + \sum_j \beta_j\right] = \frac{1}{1-\frac{1}{e}} \sum_{(i,j)\in E} x_{ij}$$

Integral online matching

Ranking: Select random total ordering π of elts of L
when new vertex in R arrives, match it to highest ranked neighbor according to π

[Karp, Vazirani, Vazirani] 1990

[Goel, Mehta] 2008

[Birnbaum, Mathieu] 2008

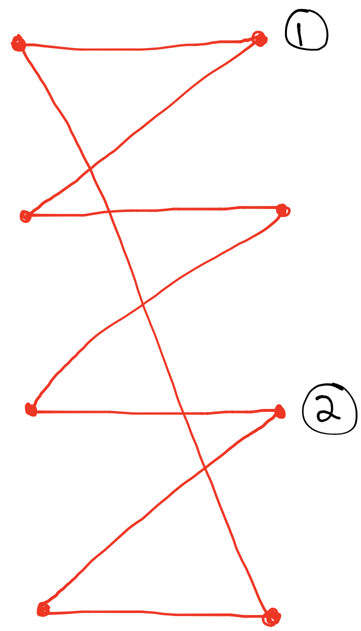
New version: [Devanur, Jain, Kleinberg] 2013

First idea: online fractional matching $1-\frac{1}{e}$

every fractional matching is convex comb of integer solutions

Specifically, construct

Online randomized rounding that constructs integral matching whose exp size = value of fractional matching produced by Water Level.



with prob $\frac{1}{2}$ ① & ② go wrong way

so exp value of integral matching $<$ value of WL

Analysis:

Reinterpretation of alg:

$\forall i \in L$ pick $Y_i \in [0,1]$ u.a.r.

when $j \in R$ arrives, match it to $\mathrm{argmin}\{Y_i \mid i \in N(j),\ i \text{ unmatched}\}$

Proof very similar to analysis we just saw!

- build feasible primal soln $P = \{x_{ij} \mid (i,j) \in E,\ x_{ij} \in \{0,1\}\}$

  and (not necessarily feasible) randomized dual

  $D = \{\alpha_i, \beta_j \mid i \in L, j \in R\}$

  such that

  (1) $P \geq c D$ <u>always</u>

  (2) $E(D)$ feasible

If so, $E(P) \geq c\ \mathrm{OPT}$

Proof: $P \geq cD \implies E(P) \geq c\ E(D) \underset{\substack{\uparrow \\ \text{feasibility of } E(D)}}{\geq} c\ \mathrm{OPT}$

To construct duals, use $g:[0,1]\to[0,1]$ $g\uparrow$ $g(1)=1$

when $x_{ij} := 1$ set $\alpha_i = \frac{g(Y_i)}{c}$ $\beta_j = \frac{1-g(Y_i)}{c}$

$\forall$ unmatched i,j set $\alpha_i := 0$, $\beta_j := 0$

$\Rightarrow$ j gets matched to i $\Delta P = 1$ $\Delta D = \frac{1}{c}$

Just need to show $E(D)$ is feasible.

Consider edge (i,j) Fix Y_{-i}

Consider execution on G_{-i} : let $y^{-i} = \begin{cases} 1 & j \text{ not matched} \\ Y_{i'} & j \text{ matched to } i' \end{cases}$

$$\beta_j^{-i} = \frac{1-g(Y_{i'})}{c}$$

Lemma

1. In actual execution $\beta_j \geq \beta_j^{-i}$

2. In actual execution, i matched if $Y_i < y^{-i}$

$\boxed{\text{Lemma} \Rightarrow \forall Y_{-i} \text{ fixed } \forall (i,j) \in E \quad E_{Y_i}[\alpha_i + \beta_j] \geq 1}$

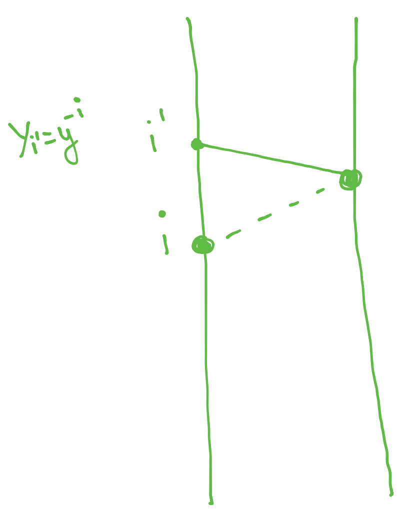
$$E_{y_i}[\alpha_i] \geq \frac{\int_0^{y} g(y)\,dy}{c} \qquad \beta_j \geq \beta_j^{-i} = \frac{1-g(y^{-i})}{c}$$

For $g(y) = e^{y-1}$ we know & love

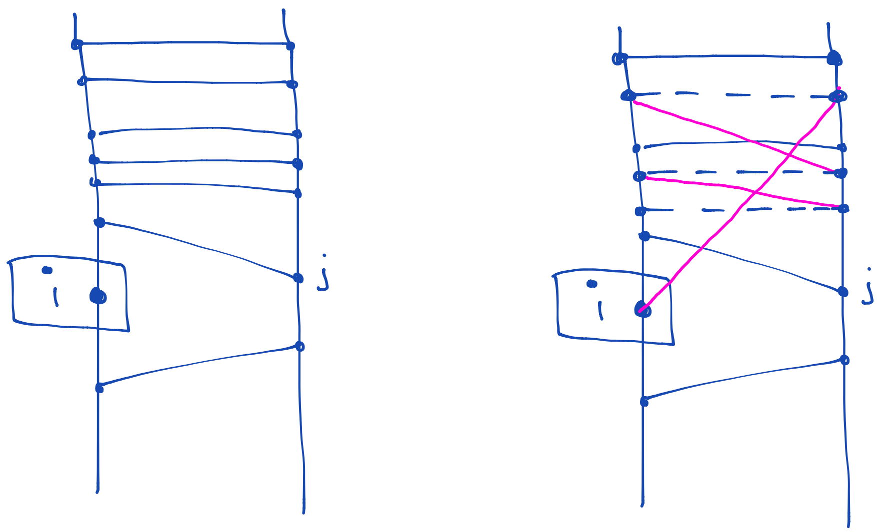
$$E_{y_i}(\alpha_i + \beta_j) \geq \frac{G(y^{-i}) - G(0) + 1 - g(y^{-i})}{c} \geq 1$$

with $c = 1 - \frac{1}{e}$

Proof of Lemma: In actual execution i matched if $Y_i < y^{-i}$



Proof of Lemma: In actual execution, $\beta_j \ge \beta_j^{-i}$



j gets matched to $\tilde{i}$ with

$y_{\tilde{i}} \le y^{-i}$

$\Rightarrow \quad \beta_j \ge \beta_j^{-i}$