Randomized Rounding of LPs

important technique for obtaining approx algs for NP-hard probs

Recipe:

1. Set problem up as integer program   vars $\in \{0,1\}$
2. Relax IP $\longrightarrow$ LP   vars $\in [0,1]$
3. Solve LP to optimality $\longrightarrow$ x_i^* $i = 1..n$
4. Construct soln by randomly rounding vars $\longrightarrow \{0,1\}$

   treat x_i^* as a probability
5. bound quality of soln by comparing to LP opt.

Example: Set Cover

Given $E = \{e_1, \ldots, e_n\}$

subsets $S_1, \ldots, S_m$ $S_j \subseteq E$

$w_1, \ldots, w_m$ w_j is "weight" of set S_j

Find $I \subseteq \{1..m\}$ s.t. $\bigcup_{j \in I} S_j = E$

and $\sum_{j \in I} w_j$ minimized

NP-complete. $\exists\, c>0$ s.t. if $\exists$ $c \ln n$

alg for Set Cover, then $P = NP$

1. Formulate Set Cover as IP

var z_j for each set S_j $\qquad z_j = \begin{cases} 1 & \text{if } S_j \text{ in cover} \\ 0 & \text{o.w.} \end{cases}$

$$\min \sum_{j=1}^{m} w_j z_j$$

$$\sum_{j \mid e_i \in S_j} z_j \geq 1 \qquad i = 1..n$$

$$z_j \in \{0,1\} \qquad j = 1..m$$

Denote by $\boxed{\text{OPT}}$ solution to this IP

2. Relax to LP

$$\min \sum_{j=1}^{m} w_j z_j$$

$$\sum_{j \mid e_i \in S_j} z_j \geq 1 \qquad i = 1..n$$

$$\cancel{z_j \in \{0,1\}} \qquad j = 1..m \quad \rightarrow \quad 0 \leq z_j \leq 1 \quad \forall j = 1..m$$

3. Solve LP

Let $z_1^*, \ldots, z_m^*$ soln

$$\boxed{V^* = \sum_{j=1}^m w_j z_j^*}$$

Observation: $V^* \le OPT$

(4) Round soln:

Include S_j with prob z_j^*

Let $Z_j = \begin{cases} 1 & S_j \text{ selected} \\ 0 & \text{o.w.} \end{cases}$ $\quad E(Z_j) = z_j^*$

$$E\left(\sum_{j=1}^m w_j Z_j\right) = \sum_{j=1}^m w_j z_j^* = V^* \le OPT \quad \text{yay}$$

Problem: result may not be set cover

$$Pr(e_i \text{ not covered}) = \prod_{j \mid e_i \in S_j} (1 - z_j^*)$$

$$\le e^{-\sum_{j \mid e_i \in S_j} z_j^*} \le \frac{1}{e}$$

const fraction of elts uncovered.

Fix by repetition

Repeat whole process k times

$$\Pr(e_i \text{ not covered}) \le \frac{1}{e^k}$$

choose $k = c \ln n$

$$\le \frac{1}{n^c}$$

$$\Pr(\exists e_i \text{ not covered}) = \Pr(e_1 \text{ not covered} \vee e_2 \text{ not covered} \vee \ldots \vee e_n \text{ not covered})$$

$$\le \sum_{i=1}^{n} \Pr(e_i \text{ not covered})$$

$$\le n \frac{1}{n^c} \le \frac{1}{n} \quad \text{for } c = 2$$

$$E(\text{cost of resulting collection of sets})$$

$$= E\left(\sum_{\ell=1}^{2\ln n} \sum_{j=1}^{m} w_j Z_j^\ell\right) = 2\ln n \sum_{j=1}^{m} w_j z_j^* = 2\ln n \, V^* \le 2\ln n \text{ OPT}$$

$$Z_j^\ell = \begin{cases} 1 & S_j \text{ selected on } \ell^{th} \text{ repetition} \\ 0 & \text{o.w.} \end{cases}$$

Result: Randomized alg which produces set cover with

expected cost $\leq c \ln n$ OPT with prob $\geq 1 - \frac{1}{n}$

$E(C \mid \text{set cover}) = ?$

$E(C) = E(C \mid \text{Set Cover}) \Pr(\text{Set Cover}) + E(C \mid \neg \text{Set Cover}) \Pr(\neg \text{Set Cover})$

$E(C \mid \text{Set Cover}) \Pr(\text{Set Cover}) = E(C) - E(C \mid \neg \text{Set Cov}) \Pr(\neg SC)$

$E(C \mid SC) \Pr(SC) \leq E(C) \quad \Rightarrow \quad E(C \mid SC) \leq \frac{E(C)}{P(SC)} \leq \frac{2 \ln n \, V^*}{1 - \frac{1}{n}} \leq 4 \ln n \, V^*$

Not so fast....

MAX SAT

Given boolean vars $x_1, \dots, x_n$

clauses $C_1, \dots, C_m$ each clause is disjunction of literals

weights $w_1, \dots, w_m$ e.g. $x_3 \vee \bar{x}_5 \vee x_{11}$

Problem: find T/F setting for vars that

maximizes $\sum_{\substack{j \text{ s.t. } C_j \\ \text{satisfied}}} w_j$

Randomized Rounding Approach:

① Formulate as IP

$$\max \sum_{j=1}^{m} w_j z_j$$

$$\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j \quad \forall\, 1 \leq j \leq m$$

P_j vars that occur +'ly

N_j vars that occur −ly

$$y_i \in \{0,1\} \ \forall\, 1 \leq i \leq n$$

$$z_j \in \{0,1\} \ \forall\, 1 \leq j \leq m$$

② Relax to LP $\quad 0 \leq y_i \leq 1 \,; \quad 0 \leq z_j \leq 1$

③ Solve LP $\rightarrow y_i^*, z_j^*$ $\quad V^*$ value of opt LP soln

④ Round $\Rightarrow$ Set $x_i = \begin{cases} T & \text{w.p. } y_i^* \\ F & \text{otherwis} \end{cases}$

Thm: Rand Rounding gives $1-\frac{1}{e}$ approx

$$\boxed{\text{Exp wt of result} = \sum_{j=1}^{m} w_j \Pr(C_j \text{ satisfied})}$$

$$\Pr(\text{clause } C_j \text{ not satisfied}) = \prod_{i \in P_j} (1-y_i^*) \prod_{i \in N_j} y_i^*$$

l_j = # literals in C_j

Arithmetic mean / geometric mean inequality $\longrightarrow$ $\leq \left[\frac{1}{l_j}\left(\sum_{i \in P_j}(1-y_i^*) + \sum_{i \in N_j} y_i^*\right)\right]^{l_j}$

$$\left(\prod_{i=1}^{k} a_i\right)^{1/k} \leq \frac{1}{k}\sum_{i=1}^{k} a_i \quad \forall \text{ nonnegative } a_i$$

$$= \left[1 - \frac{1}{l_j}\underbrace{\left(\sum_{i \in P_j} y_i^* + \sum_{i \in N_j}(1-y_i^*)\right)}_{\geq z_j^*}\right]^{l_j}$$

$$\leq \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$$

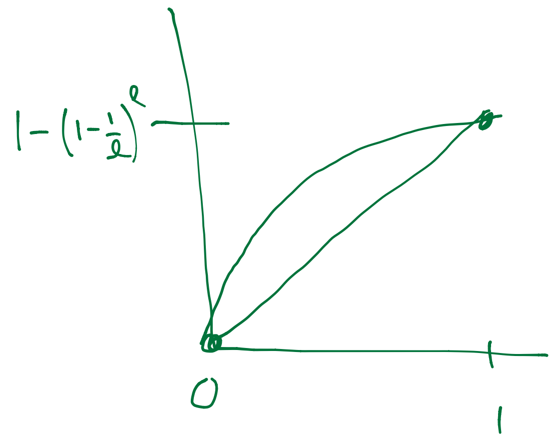
$$\Pr(C_j \text{ is satisfied}) \geq 1 - \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$$

$$\geq \left[1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right] z_j^*$$

$f(x) = 1 - \left(1 - \frac{x}{\ell}\right)^{\ell}$ concave in $[0,1]$ for $\ell \geq 1$

$f'(x) = -\ell\left(1 - \frac{x}{\ell}\right)^{\ell-1}\left(-\frac{1}{\ell}\right) = \left(1 - \frac{x}{\ell}\right)^{\ell-1}$

$f''(x) = (\ell-1)\left(1 - \frac{x}{\ell}\right)^{\ell-2}\left(-\frac{1}{\ell}\right) \leq 0$ on $x \in [0,1]$



$1 - \left(1 - \frac{x}{\ell}\right)^{\ell} \geq \left[1 - \left(1 - \frac{1}{\ell}\right)^{\ell}\right] x$

$x \in [0,1]$

Putting it all together

$$\text{Exp wt of result} \geq \sum_{j=1}^{m} w_j z_j^* \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right]$$

$$\geq V^* \min_{k \geq 1} \left[1 - \underbrace{\left(1 - \frac{1}{k}\right)^{k}}\right]$$

$\left(1 - \frac{1}{k}\right)^{k} \leq e^{-1}$ Most useful approx ever

$\longrightarrow \uparrow$ fn of k

$$\geq V^*\left(1 - \frac{1}{e}\right)$$

$\square$

Can we do better? Do at end.

One kind of limit: integrality gap.

$$\text{Integrality gap of IP} = \inf_{\text{instances } I} \frac{\text{IP optimum of } I}{\text{LP optimum of } I}$$

Example: $x_1 \vee x_2$, $x_1 \vee \bar{x}_2$, $\bar{x}_1 \vee x_2$, $\bar{x}_1 \vee \bar{x}_2$ all weights $= 1$

IP OPT $= 3$

$\Rightarrow IG \leq \frac{3}{4}$

LP OPT $= 4$

If we get our bound by comparing value of soln to to LP OPT, can't hope to get approx ratio better than IG.

Let's shoot for $\frac{3}{4}$

we'll do this with non-linear rounding

not required to set x_i T w.p. y_i^*

could set x_i T w.p. $f(y_i^*)$ $\qquad f: [0,1] \to [0,1]$

Want $Pr(\text{clause } C_j \text{ satisfied}) \geq 1 - 4^{-z_j^*} \geq \left(1 - \frac{1}{4}\right) z_j^* = \frac{3}{4} z_j^*$

$1 - 4^{-x}$ concave in $[0,1]$

$$Pr(\text{clause } C_j \text{ not satisfied}) = \prod_{i \in P_j} (1 - f(y_i^*)) \prod_{i \in N_j} f(y_i^*)$$

$$\leq 4^{-\left[\sum_{i \in P_j} y_i^* + \sum_{i \in N_j} (1 - y_i^*)\right]} = 4^{-z_j^*}$$

need $\quad 1 - f(x) \leq 4^{-x} \quad$ and $\quad f(x) \leq 4^{-(1-x)}$

i.e $\quad 1 - 4^{-x} \leq f(x) \leq 4^{x-1}$

this is feasible $\quad 1 - 4^{-x} \leq 4^{x-1} \quad$ with equality at $x = \frac{1}{2}$

$\in [0,1]$

With this choice $E(\text{wt of soln}) = \sum_{j=1}^{m} w_j \Pr(\text{clause } j \text{ satisfied})$

$$\geq \frac{3}{4} \sum_{j=1}^{m} w_j z_j^* \geq \frac{3}{4} \text{OPT}$$