CSE 525 Randomized Algorithms & Probabilistic Analysis **Spring 2013**

Lecture 15: May 23

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15.1 General Random Walks

In previous lectures we have discussed random walks on Markov chains $G = (V, E)$. Now we want to extend the notion of random walks to graphs where each edge has a **weight**. Because we wish to set up a connection to electrical networks, we will call this nonnegative weight the **conductance**. So for an edge e_{xy}, we notate its conductance c_{xy}. The intuition is that edges with higher conductance have a higher probability of being chosen in a random walk. Specifically,

$$P_{xy} = \frac{c_{xy}}{c_x}$$

where P_{xy} is the probability of taking edge e_{xy} from x in our random walk, and c_x is defined as $\sum_{y_i|\{x,y_i\}\in E} c_{xy_i}$, thus representing the overall conductance of the vertex x.

It is easy to verify that the stationary distribution is as follows:

$$\pi_x = \frac{c_x}{c_G}$$

where $c_G = \sum_{x\in V} c_x$. This is verified using the fact we showed from last time that $\pi_x = \pi P$, where P is the transition matrix of the markov chain.

15.2 Harmonic extensions

We call a function $h : V \mapsto \mathbb{R}$ **harmonic** with respect to some Markov Chain if

$$h(i) = \sum_{j|\{i,j\}\in E} P_{ij} h(j)$$

Intuitively, a function over verticies is harmonic if it is a weighted average of all neighboring verticies (where the weights correspond to the probability of moving to that neighbor).

Now, imagine we have a subset of verticies $B \subset V$ (we will sometimes call this a **boundary**), and a function over that subset $h_B : B \mapsto \mathbb{R}$. Then a **harmonic extension** of h_B is a function such that:

1. $h(x) = h_B(x) \forall x \in B$
2. h is harmonic on all $x \notin B$

So given an arbitrary subset of vertices and an arbitrary function over that subset, a harmonic extension matches that function on the subset, and for verticies not in that subset "propagates out" the function values in a harmonic fashion.

Proposition 15.1 *Given an irreducible markov chain, $B \subset V$, and $h_B : B \mapsto \mathbb{R}$, the function over verticies*

$$h(x) = \mathbb{E}\left[h_B(X_{T_B})\right]$$

is the unique harmonic extension of h_B.

What this is saying is this: Imagine starting at a vertex x and doing a random walk until you reach the first vertex in B. T_B denotes the time that you first hit B. X_{T_B} denotes that first vertex you hit in B. And $h_B(X_{T_B})$ is the function value at that vertex. So we claim that the expected value of this function is a harmonic extension of h_B.

It may be helpful to think about the case where $h_B(x)$ is the same value v for all $x \in B$. In that case, $\mathbb{E}\left[h_B(X_{T_B})\right] = v$ as well, since whenever we hit B we always get value v. So the only interesting case is where different verticies in $x \in B$ have different $h_B(x)$.

15.2.1 h is a harmonic extension

Now, we formally prove Proposition 15.1 - specifically that $h(x)$ is a harmonic extension.

Proof: First consider the case where $x \in B$. In this case $T_B = 0$ so

$$h(x) = \mathbb{E}\left[h_B(x)\right] = h_B(x)$$

Satisfying part 1 of the definition of a harmonic extension.

Now, what if $x \notin B$? In that case:

$$h(x) = \mathbb{E}\left[h_B(X_{T_B})\right] = \sum_{y|\{x,y\}\in E} P_{xy}\, \mathbb{E}\left[h_B | x_1 = y\right]$$

by the definition of a random walk. However, since the chain is Markov, and hence time-homogeneous (i.e. memoryless): $\mathbb{E}\left[h_B | x_1 = y\right] = \mathbb{E}\left[h_B | x_0 = y\right] = h(y)$. Hence $h(x) = \sum_{y|\{x,y\}\in E} P_{xy} h(y)$, as required for the harmonic property. ∎

15.2.2 Uniqueness

So we have shown that $h(x)$ is a harmonic extension, but we have not shown that this extension is unique. We will now prove uniqueness.

Proof: The intuition for our function's uniqueness comes from thinking about this extension inductively, where B is the base case and the inductive step propogates the value from a vertex to its neighbors, resulting in a single solution.

To more formally show this, imagine we have two harmonic extensions, h and g. Now, we know for all $x \in B$, $h(x) = h_B(x) = g(x)$, so $h(x) - g(x) = 0$.

So take the vertex $x_i \notin B$ where $h(x) - g(x)$ is maximized. Since both h and g are harmonic, the value of $h(x_i) - g(x_i)$ is the weighted average of the $h(x_j) - g(x_j)$ for each $j|\{i,j\} \in E$. Because $h(x_i) - g(x_i)$ is the global maximum, the only way to have the average work out is for $h(x_j) - g(x_j) = v = h(x_i) - g(x_i)$ (if one was higher, it would not be the max, if one were lower, the weighted average would not work). We can keep applying this argument to say all neighbors of every j also have to have value v, etc. But since there is a path from x_i to some $b \in B$, b must also have value v, but since we know that $v = h(b) - g(b) = 0$, the maximum difference in values is zero.

For completeness, we also have to show that the minimum value is zero. Take the vertex $x_i \notin B$ where $h(x) - g(x)$ is minimized. Since both h and g are harmonic, the value of $h(x_i) - g(x_i)$ is the weighted average of the $h(x_j) - g(x_j)$ for each $j|\{i,j\} \in E$. Because $h(x_i) - g(x_i)$ is the global minimum, the only way to have the average work out is for $h(x_j) - g(x_j) = v = h(x_i) - g(x_i)$. We can keep applying this argument to say all neighbors of every j also have to have value v, etc. But since there is a path from x_i to some $b \in B$, b must also have value v, but since we know there $v = h(b) - g(b) = 0$, the minimum difference in values is zero.

Thus, it follows that the $g(x) = h(x)$ for all nodes x in our chain. As such, our two extension functions h and g must be equal, and thus there may only be one, unique harmonic extension for any set B and function h_B. ■

15.3 Electrical Networks, Voltages, and Current Flows

We now extend our understanding of weighted Markov Chains to the application of electrical networks.

Note that we will continue to refer to our electrical network as a Markov Chain/graph described by $G = (V, E)$. We also introduce the following notation to describe our edges:

$\vec{e}_{xy}$
: The *directed* edge, traveling from x to y

We now choose two nodes s and t to be our source and sink nodes, respectively. That is, we will discuss the properties of an electrical current flowing from our source s through our network, to sink t.

Relating back to our discussion of harmonic extensions, pick a set $B = \{s, t\}$. Our "harmonic function" will be denoted $\phi(x)$, and will signify the **voltage** of a given node. Hence, by the uniqueness property of any harmonic extension, we get that the voltage $\phi(x)$ of every node $x \notin \{s, t\}$ is completely determined by $\phi(s)$ and $\phi(t)$.

15.3.1 Properties of Flow

We now discuss the properties of a general flow through a network. A flow is a function f over *directed* edges, such that:

1. f is anti-symmetric, that is:
$$f(\vec{e}) = -f(\overleftarrow{e})$$
2. The flow into any node $x \notin \{s, t\}$ is equal to the flow out of the node. Anti-symmetry thus means that the total flow over all edges incident to x will sum to 0. More formally:
$$\forall x \notin \{s, t\} : \sum_{y|\{x,y\} \in E} f(\vec{e}_{xy}) = 0$$

(this is known as **Kirchoff's Law**)

3. A non-negative amount of flow must come out of s. That is:

$$\sum_{y|\{s,y\}\in E} f(\vec{e}_{sy}) \geq 0$$

This value is the "strength" of the flow.

From these properties, we get the following as a free corollary:

Corollary 15.2

$$\sum_{x} \sum_{y|\{x,y\}\in E} f(\vec{e}_{xy}) = 0$$

That is, the net sum of flow over every directed edge is 0.

Proof: This follows directly from properties 2 and 3. That is, property 2 tells us that the net flow over all nodes $x \notin \{s,t\}$ is 0. Next, from property 3 we know that the net flow out of node s is non-negative. This flow travels through the network, and must end up at sink t, in the same quantity (as property 2 guarantees that no flow may be lost along the way). Thus, the flow for node t will be equal in magnitude, but opposite in sign of that of node s. Therefore, these flows will cancel, leaving us with 0 net flow for the entire network. ■

15.3.2 Electrical Flow

With these properties of flow in place, we go about constructing an electrical network by, placing a resistor on each edge e, such that the **resistance** r_e on e satisfies the following:

$$r_e = \frac{1}{c_e}$$

(that is, the resistance on an edge is inversely proportional to its conductance)

So, given the voltages ϕ on every edge, we may find the flow of a current - denoted I - on any edge by the following property:

Definition 15.3

$$I(\vec{e}_{xy}) = \frac{\phi(x) - \phi(y)}{r_{xy}}$$

Our definition of resistance yields the following useful form for finding an edge's current:

Observation 15.4

$$I(\vec{e}_{xy}) = c_e(\phi(x) - \phi(y))$$

This property of current is known as **Ohm's Law**. From this law, we may make the following three observations about any electrical current:

Observation 15.5

$$\forall x \notin \{s,t\} \sum_{y|(x,y)\in E} I(\vec{e}_{xy}) = 0$$

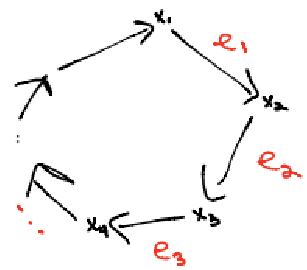

Figure 15.1:

That is, current is conserved (essentially showing that Kirchoff's Law holds for our flow of electrical current).

Proof:

$$\begin{aligned}
\sum_{y|(x,y)\in E} I(\vec{e}_{xy}) &= \sum_{y|(x,y)\in E} c_{xy}(\phi(x)-\phi(y)) && [\textit{by definition}]\\
&= \sum_{y|(x,y)\in E} c_{xy}\phi(x) - \sum_{y|(x,y)\in E} c_{xy}\phi(y)\\
&= \phi(x) \sum_{y|(x,y)\in E} - \sum_{y|(x,y)\in E} c_{xy}\phi(y)\\
&= \phi(x)c_x - \sum_{y|(x,y)\in E} c_{xy}\phi(y)\\
&= c_x[\phi(x) - \sum_{y|(x,y)\in E} \frac{c_{xy}}{c_x}\phi(y)]\\
&= c_x[\phi(x) - \sum_{y|(x,y)\in E} p_{xy}\phi(y)]\\
&= c_x[\phi(x)-\phi(x)] && [\textit{by harmonicity}]\\
&= 0
\end{aligned}$$

■

Observation 15.6 *Suppose we have a cycle of n edges (and thus n vertices), such as the cycle in 15.1. If we impose ordering $e_1, ..., e_n$ of the edges and $x_1, ..., x_n$ on the vertices in sequence around the cycle (as is done in 15.1), then:*

$$\sum_i r_{\vec{e}_i} I(\vec{e}_i) = 0$$

That is, the sum of the potential differences (i.e. difference in voltages) across each edge in a cycle is 0. Note that this is referred to as the **Cycle Law**. This should intuitively follow from Kirchoff's Law, but the formal proof is as follows:

Proof:

$$\sum_i r_{\vec{e}_i} I(\vec{e}_i) = \sum_i \phi(x_i) - \phi(x_{i+1})$$
$$= 0$$

The final step in our proof comes from recognizing that we have a telescoping sum, in which all terms will cancel with themselves.

Note also that the orientation of travel around the cycle is irrelevant - current is anti-symmetric, so traveling in the opposite direction will simply reverse the sign of each term in our summation, and still come out to 0. ■

Observation 15.7 *For some value c, consider the following voltage functions ϕ and ϕ', such that:*

$$\forall x : \phi'(x) = \phi(x) + c$$

Then $\phi'(x)$ shares the same current and harmonicity of $\phi(x)$.

Proof: Let I be the current flow of $\phi(x)$, and I' the current of $\phi'(x)$ Then for any edge e_{xy}:

$$\begin{aligned} I'(\vec{e}_{xy}) &= \frac{\phi'(x) - \phi'(y)}{r_{xy}} \\ &= \frac{\phi(x) + c - (\phi(y) + c)}{r_{xy}} \\ &= \frac{\phi(x) - \phi(y)}{r_{xy}} \\ &= I(\vec{e}_{xy}) \end{aligned}$$

Thus we see that the current across any edge will remain the same, if the voltage function is increased by a constant factor for all nodes. Further, we recall that, in order for a function to be harmonic, the following must hold:

$$\phi(i) = \sum_{j|\{i,j\}\in E} P_{ij}\phi(j)$$

Well, it is quite clear to see that ϕ' is still harmonic, if ϕ was harmonic in the first place. That is:

$$\begin{aligned} \phi'(i) &= \sum_{j|\{i,j\}\in E} P_{ij}\phi'(j) \\ &= \sum_{j|\{i,j\}\in E} P_{ij}(\phi(j) + c) \\ &= \sum_{j|\{i,j\}\in E} P_{ij}\ \phi(j) + \sum_{j|\{i,j\}\in E} P_{ij}c \\ &= [\sum_{j|\{i,j\}\in E} P_{ij}\ \phi(j)] + c \sum_{j|\{i,j\}\in E} P_{ij} \\ &= [\sum_{j|\{i,j\}\in E} P_{ij}\ \phi(j)] + c && [\sum P_{ij} = 1] \\ &= \phi(i) + c \end{aligned}$$

(the intuition here being that, if the voltages of all nodes incident to a given node x are each increased by c, then the voltage of x will still be the average of those around, and thus harmonic - but only c units larger!)

We have thus shown that adding a constant to our voltage function does not affect current, nor the harmonicity of the function, thereby completing our proof. ■

An important result from our third observation is that we may in a sense "scale" up or down our voltage function by any additive constant, without changing any properties of the network. As such, WLOG we may always set $\phi(t) = 0$ (i.e. we may take any satisfying ϕ function, and scale down with an additive constant to force $\phi(t) = 0$). This therefore means that our voltage function for all non-source/sink nodes in our graph is determined entirely by the value of $\phi(s)$ (because our harmonic extension is unique, and based entirely upon our set $B = \{s, t\}$, and $\phi(t)$ is now "fixed").

To tie together our discussion of general flows with current flows on electrical networks, we note the following property:

Theorem 15.8 *Suppose f is some satisfying flow on our network. Then, if*

1. *f satisfies the Cycle Law*
2. $\sum_{y|(s,y)\in E} f(\vec{e}_{s,y}) = \sum_{y|(s,y)\in E} I(\vec{e}_{s,y})$
   (i.e. the total flow out of s is equal in both f and the electrical current flow I)

then it follows that our flow f = the current flow I on every edge.

15.3.3 Effective Resistance

An important concept in electrical networks is the amount of resistance between two nodes x and y given by the entire network. We call this the **Effective Resistance**, and it essentially behaves as if we were to treat the entire network as one large resistor. This resistance is notated $R(x \Leftrightarrow y)$, and can be found as follows:

Definition 15.9

$$R(x \Leftrightarrow y) = \frac{\phi(x) - \phi(y)}{||I||}$$

Where $||I||$ represents the **strength** or **value** of the flow (i.e. the total flow out of s). That is:

$$||I|| = \sum_{y|(s,y)\in E} I(\vec{e}_{sy})$$

Given this definition of effective resistance, we note the following:

Observation 15.10 *The effective resistance $R(s \Leftrightarrow t)$ between our source and sink nodes s and t is completely independent of our value of ϕ, and is thus just a property of the graph itself.*

Proof: Consider two different, but valid voltage functions ϕ and ϕ' on the same electrical network G. WLOG, we may "scale down" both functions such that $\phi(t) = 0$ and $\phi'(t) = 0$. Following this scale-down, suppose the following is true for some value c:

$$\phi'(s) = c * \phi(s)$$

As such:

$$R(s \Leftrightarrow t) = \frac{\phi(s) - \phi(t)}{||I||} = \frac{\phi(s)}{||I||}$$

and

$$R'(s \Leftrightarrow t) = \frac{\phi'(s) - \phi'(t)}{||I'||} = \frac{\phi'(s)}{||I'||}$$

We note that $||I||$ represents the flow value of a given current, which by definition is the total flow out of s under the voltage function ϕ.

Now, we must simply show that $||I'|| = c * ||I||$. In order to show this, first consider the original voltage function ϕ. If we were to multiply *all* voltages by the value c, then note that our voltage function is still a valid harmonic function (that is, every $\phi(x)$ would still be the average of all of its neighbors - this average would just be scaled up by a multiplicative factor c). Further, we note that in this new harmonic function, our voltage at s is $c * \phi(s)$, and we still have $\phi(t) = 0$.

We now recall that our originally discussed function ϕ' had the property that $\phi'(s) = c * \phi(s)$, and $\phi'(t) = 0$, as is the case in the (scaled) harmonic function we just derived above. Because harmonic extensions are unique, it must be the case that ϕ' is precisely this harmonic function! That is, for all x in our network, we now know that $\phi'(x) = c * \phi(x)$. Given that this is the case, we may find the value of $||I'||$ as follows:

$$\begin{aligned}
||I'|| &= \sum_{y|(s,y)\in E} I'(\vec{e}_{sy}) \\
&= \sum_{y|(s,y)\in E} c_{sy}(\phi'(s) - \phi'(y)) \\
&= \sum_{y|(s,y)\in E} c_{sy}(c\phi(s) - c\phi(y)) \\
&= \sum_{y|(s,y)\in E} c * c_{sy}(\phi(s) - \phi(y)) \\
&= \sum_{y|(s,y)\in E} c * I(\vec{e}_{sy}) \\
&= c \sum_{y|(s,y)\in E} I(\vec{e}_{sy}) \\
&= c * ||I||
\end{aligned}$$

Thus, we now have the following:

$$\begin{aligned}
R'(s \Leftrightarrow t) &= \frac{\phi'(s)}{||I'||} \\
&= \frac{c * \phi(s)}{c * ||I||} \\
&= \frac{\phi(s)}{||I||} \\
&= R(s \Leftrightarrow t)
\end{aligned}$$

We have thereby completed our proof and hence, shown that the effective resistance between the source and sink nodes is completely independent of the voltages placed on each node, and rather is a property of the resistances and the network itself. ∎

15.4 Random Walks on Electrical Networks

We now relate the discussed properties of electrical networks back to our notion of a random walk on Markov Chains (as these electrical networks are in fact Markov Chains themselves).

15.4.1 Three Lemmas

The following three lemmas relate to random walks on our Markov Chains. They are not only substantial in themselves, but also form the basis of theorems presented later.

15.4.2 Lemma 1: Probability of hitting T before S

Lemma 15.11 *Let T_{st} denote the time (i.e. number of steps) it takes, starting at s, to reach t, and T_{ss}^{+} denote the time it takes, starting at s, to leave s and return. Then,*

$$Pr(T_{st} < T_{ss}^{+}) = \frac{1}{c_s R(s \Leftrightarrow t)}$$

Intuitively this means that if the effective resistance is low (meaning there are lots of high-conductance paths from s to t), then the probability of reaching t before returning to s will be high.

Proof: The function $g(x) = Pr(T_{xt} < T_{xs})$ is a harmonic function over $V/\{s,t\}$, since the probability of a random walk ending up at s or t is calculated as a weighted average on the neighbors of x. The boundary is $\{s,t\}$, and we fix $g(s) = 0$ (since if $x = s$ we must have hit s again) and $g(t) = 1$ (since if $x = t$ we hit t before returning to s).

Now, take the function:

$$h(x) = \frac{\phi(s) - \phi(x)}{\phi(s) - \phi(t)}$$

For a harmonic extension ϕ with boundary set $\{s,t\}$. Globally adding u and and scaling by w a harmonic function will cause it to remain harmonic, since

$$wh_0(i) + u = u + w \sum_{j|\{i,j\} \in E} P_{ij} h_0(j) = \sum_{j|\{i,j\} \in E} P_{ij}(wh_0(j) + u)$$

so, $h(x)$ is still harmonic on $V/\{s,t\}$. And, $h(s) = 0 = g(s)$ and $h(t) = 1 = g(t)$.

Therefore, h and g are both harmonic extensions with the same boundary values. So, by proposition 15.1, $h(x) = g(x) = Pr(T_{xt} < T_{xs})$.

So now unrolling the desired quantity one step,

$$\begin{aligned}
Pr(T_{st} < T_{ss}^{+}) &= \sum_x P_{sx} Pr(T_{xt} < T_{xs}) \\
&= \sum_x \frac{c_{sx}}{c_s} h(x) \\
&= \sum_x \frac{c_{sx}}{c_s} \frac{\phi(s) - \phi(x)}{\phi(s) - \phi(t)} \\
&= \sum_x \frac{I(s \mapsto x)}{c_s(\phi(s) - \phi(t))} \\
&= \frac{\sum_x I(s \mapsto x)}{c_s(\phi(s) - \phi(t))} \\
&= \frac{||I||}{c_s(\phi(s) - \phi(t))} \\
&= \frac{1}{c_s R(s \Leftrightarrow t)}
\end{aligned}$$

■

15.4.3 Lemma 2: Expected number of times to hit S before T

Lemma 15.12 $\mathbb{E}\left[\#\ \textit{of visits to s before t}\right] = c_s R(s \Leftrightarrow t)$

Proof: From Lemma 15.11 we know that $Pr(T_{st} < T_{ss}^{+}) = \frac{1}{c_s R(s \Leftrightarrow t)}$. Now, each time a random walk returns to S before hitting T, we can start the random walk again again and the probability of hitting T before S remains the same.

So we can think of random walking from S to T as a sequence of Bernoulli trials, where in each trial there is probability $p = \frac{1}{c_s R(s \Leftrightarrow t)}$ of hitting t before S, and $1 - p$ of hitting S before T. Then the number of Bernoulli trials until we hit T is distributed as a Geometric random variable with parameter p. And we know that $\mathbb{E}\left[Geom(p)\right] = \frac{1}{p}$. So $\mathbb{E}\left[\#\text{ of visits to s before t}\right] = c_s R(s \Leftrightarrow t)$. ■

15.4.4 Lemma 3: Expected number of times to hit x before stopping

First we need to define the term **stopping time**. Any deterministic $f_\tau : X_0 \dots X_t \mapsto \{0, 1\}$ is a stopping function, where 1 indicates we have stopped and 0 indicates we continue. Note that f_τ can depend only on the history of states. For example, a possible valid stopping function might stop if and only if we have seen X_0 seven times, and X_2 exactly twice as much as X_1. An invalid example would be stopping after the last time we visit S in the first 100 steps (because it requires knowledge of the future to determine if we can stop at time t).

Call τ the event that f_τ returns 1, meaning we have stopped.

Lemma 15.13 *For any f_τ such that $Pr(X_\tau = S) = 1$, for all x,*

$$\frac{\mathbb{E}\left[\#\ of\ visits\ to\ x\ before\ \tau\right]}{\mathbb{E}\left[\tau\right]} = \pi_x$$

You are asked to show this in HW 4, so we will not give the proof here.

15.4.5 Commute Time Identity

The following identity describes the expected time required to travel from the source s to the sink t, and back, on a random walk over our electrical network. That is, consider a random walk on graph G, then:

Theorem 15.14

$$\mathbb{E}[T_{st}] + \mathbb{E}[T_{ts}] = c_G R(s \Leftrightarrow t)$$

(recalling that T_{st} denotes the time taken to travel from s to t)

Proof:

$$\text{let stopping time } \tau = T_{st} + T_{ts}$$

We are thus looking to find $\mathbb{E}[\tau]$. Fortunately, Lemma 3 from above allows us to do just that, via the following equation:

$$\frac{\mathbb{E}[\# \ of \ visits \ to \ s \ before \ \tau]}{\mathbb{E}[\tau]} = \pi_s$$

Recalling from earlier our equation for π_x, we get the following:

$$\frac{\mathbb{E}[\# \ of \ visits \ to \ s \ before \ time \ \tau]}{\mathbb{E}[\tau]} = \frac{c_s}{c_G}$$

Further, we note that our numerator

$$\mathbb{E}[\# \ of \ visits \ to \ s \ before \ \tau]$$

is semantically equivalent to

$$\mathbb{E}[\# \ of \ visits \ to \ s \ before \ reaching \ t]$$

because all visits to s before τ must also occur before reaching t (i.e. the first time visiting s *after* visiting t will in fact be at time τ, so the two conditions are equivalent). This new definition is described exactly by Lemma 2 from above. Thus, plugging in, we get:

$$\begin{aligned}
\frac{\mathbb{E}[\# \ of \ visits \ to \ s \ before \ reaching \ t]}{\mathbb{E}[\tau]} &= \frac{c_s}{c_G} \\
\frac{c_s R(s \Leftrightarrow t)}{\mathbb{E}[\tau]} &= \frac{c_s}{c_G} \\
c_G R(s \Leftrightarrow t) &= \mathbb{E}[\tau] \\
c_G R(s \Leftrightarrow t) &= \mathbb{E}[T_{st}] + \mathbb{E}[T_{ts}]
\end{aligned}$$

■

15.5 Series and Parallel Laws

Theorem 15.15 *Resistances in series add. Specifically, imagine we have three vertices u, v and w, where v has degree 2, connected in a line. There is a $\{u,v\}$ edge e_1 with conductance c_1 and a $\{v,w\}$ edge e_2 with conductance c_2. We can then remove v and replace the edges with a single $\{u,w\}$ edge e_n with conductance $\frac{1}{\frac{1}{c_1}+\frac{1}{c_2}} = \frac{c_1 c_2}{c_1+c_2}$. This is known as the* ***Series Law****.*

Proof: We need to prove that this will preserve both current and voltage. Let's start with voltage. Without loss of generality, we need to show that $\phi(u) = \phi'(u)$, where $\phi'(u)$ is the new harmonic extension after the edge replacement. Clearly, if u is S or T, the voltage is fixed and $\phi(u) = \phi'(u)$ by definition. So imagine that u is connected to some other set of vertices T. We can envisioning replacing the set T with a single vertex as follows: Let $c_T = \sum_{t \in T} c_t$ and $\phi(T) = \sum_{t \in T} \frac{c_t}{c_T}\phi(t)$. Now, we know:

$$\phi(u) = \frac{c_1}{c_1+c_T}\phi(v) + \frac{c_T}{c_1+c_T}\phi(T)$$

We now know that, since v is of degree 2, $\phi(v) = \frac{c_1}{c_1+c_2}\phi(u) + \frac{c_2}{c_1+c_2}\phi(w)$, so:

$$\begin{aligned}
\phi(u) &= \frac{c_1}{c_1+c_T}\left(\frac{c_1}{c_1+c_2}\phi(u) + \frac{c_2}{c_1+c_2}\phi(w)\right) + \frac{c_T}{c_1+c_T}\phi(T) \\
(c_1+c_T)\phi(u) &= c_1\left(\frac{c_1}{c_1+c_2}\phi(u) + \frac{c_2}{c_1+c_2}\phi(w)\right) + c_T\phi(T) \\
(c_1+c_T)\phi(u) &= \frac{c_1^2}{c_1+c_2}\phi(u) + \frac{c_1 c_2}{c_1+c_2}\phi(w) + c_T\phi(T) \\
\frac{(c_1+c_2)(c_1+c_T) - c_1^2}{c_1+c_2}\phi(u) &= \frac{c_1 c_2}{c_1+c_2}\phi(w) + c_T\phi(T) \\
\frac{c_T(c_1+c_2) + c_1 c_2}{c_1+c_2}\phi(u) &= \frac{c_1 c_2}{c_1+c_2}\phi(w) + c_T\phi(T) \\
\left(\frac{c_1 c_2}{c_1+c_2} + c_T\right)\phi(u) &= \frac{c_1 c_2}{c_1+c_2}\phi(w) + c_T\phi(T) \\
\phi(u) &= \frac{\frac{c_1 c_2}{c_1+c_2}}{\frac{c_1 c_2}{c_1+c_2} + c_T}\phi(w) + \frac{c_T}{\frac{c_1 c_2}{c_1+c_2} + c_T}\phi(T)
\end{aligned}$$

This is precisely the value of $\phi'(u)$ assuming that $\phi'(w) = \phi(w)$, since $c_n = \frac{c_1 c_2}{c_1+c_2}$. Since the same can be shown for w, leaving the voltages unchanged will still result in a valid solution.

Now, we need to show that, given unchanged voltages, the current on the new edge will be unchanged. By conservation of current, $I(v \mapsto w) = I(u \mapsto v) = c_1(\phi(u) - \phi(v))$. Now, $\phi(v) = \frac{c_1}{c_1+c_2}\phi(u) + \frac{c_2}{c_1+c_2}\phi(w)$, so:

$$
\begin{aligned}
I(u \mapsto v) =& c_1\phi(u) - c_1\left(\frac{c_1}{c_1+c_2}\phi(u) + \frac{c_2}{c_1+c_2}\phi(w)\right) \\
I(u \mapsto v) =& \frac{c_1(c_1+c_2)}{c_1+c_2}\phi(u) - \frac{c_1^2}{c_1+c_2}\phi(u) - \frac{c_1c_2}{c_1+c_2}\phi(w) \\
I(u \mapsto v) =& \frac{c_1^2 + c_1c_2 - c_1^2}{c_1+c_2}\phi(u) - \frac{c_1c_2}{c_1+c_2}\phi(w) \\
I(u \mapsto v) =& \frac{c_1c_2}{c_1+c_2}\phi(u) - \frac{c_1c_2}{c_1+c_2}\phi(w) \\
I(u \mapsto v) =& \frac{c_1c_2}{c_1+c_2}(\phi(u) - \phi(w))
\end{aligned}
$$

Since $c_n = \frac{c_1c_2}{c_1+c_2}$, this is precisely the current after the transformation. ■

Theorem 15.16 *Conductances in parallel add. Specifically, if we have two vertices u and v, and two $\{u, v\}$ edges e_1 and e_2 with conductances c_1 and c_2, then we can replace them with a single edge e_n with conductance $c_1 + c_2$. This is known as the* ***Parallel Law****.*

Proof:

As before, we need to show that the voltages and current stay the same. Let's start with voltage. Without loss of generality, we need to show that $\phi(u) = \phi'(u)$, where $\phi'(u)$ is the new harmonic extension after the edge replacement. Clearly, if u is S or T, the voltage is fixed and $\phi(u) = \phi'(u)$ by definition. So imagine that u is connected to some other set of vertices T. We can envisioning replacing the set T with a single vertex as follows: Let $c_T = \sum_{t\in T} c_t$ and $\phi(T) = \sum_{t\in T}\frac{c_t}{c_T}\phi(t)$. Now, we know:

$$
\phi(u) = \frac{c_1}{c_1+c_2+c_T}\phi(v) + \frac{c_2}{c_1+c_2+c_T}\phi(v) + \frac{c_T}{c_1+c_2+c_T}\phi(T)
$$

$$
= \frac{c_1+c_2}{c_1+c_2+c_T}\phi(v) + \frac{c_T}{c_1+c_2+c_T}\phi(T) = \phi'(u)
$$

Now, we need to show that, given unchanged voltages, the current on the new edge will be unchanged. Since flows add, we know that in the original graph the total flow from u to v is $I(e_1) + I(e_2) = c_1(\phi(u) - \phi(v)) + c_2(\phi(u) - \phi(v)) = (c_1 + c_2)(\phi(u) - \phi(v))$, which is identical to the current after the edge replacement.

■