CSE522 - Learning Theory - Homework Exercise 3

1. Choose 4 of the following 5 hypothesis classes and formally calculate their VC dimension:

   (a) $\mathcal{X} = \mathbb{R}$, $\mathcal{H}$ is the union of two intervals. In other words, each $h \in \mathcal{H}$ is defined by four reals $\alpha, \beta, \gamma, \delta$ as
   $$h_{\alpha,\beta,\gamma,\delta}(x) = \begin{cases} +1 & \text{if } x \in [\alpha, \beta] \cup [\gamma, \delta] \\ -1 & \text{otherwise} \end{cases} .$$

   (b) $\mathcal{X}$ is the set of all intervals on the reals, $\mathcal{H}$ is the real line. In other words, each $h \in \mathcal{H}$ is defined by a real α as
   $$h_\alpha([a, b]) = \begin{cases} +1 & \text{if } \alpha \in [a, b] \\ -1 & \text{otherwise} \end{cases} .$$

   (c) $\mathcal{X} = \mathbb{R}^2$, $\mathcal{H}$ is the set of axis-parallel squares (with equal height and width). In other words, each $h \in \mathcal{H}$ is defined by three reals α, β, γ as
   $$h_{\alpha,\beta,\gamma}(x) = \begin{cases} +1 & \text{if } x_1 \in [\alpha, \alpha + \gamma] \text{ and } x_2 \in [\beta, \beta + \gamma] \\ -1 & \text{otherwise} \end{cases} .$$

   (d) $\mathcal{X} = \mathbb{R}^2$, $\mathcal{H}$ is the set of unit balls. In other words, each $h \in \mathcal{H}$ is defined by a point $c \in \mathbb{R}^2$ as
   $$h_c(x) = \begin{cases} +1 & \text{if } \|x - c\|_2 \le 1 \\ -1 & \text{otherwise} \end{cases} .$$

   (e) $\mathcal{X}$ is the positive integers, $\mathcal{H}$ has 7 functions: $h_1, h_2, h_3, h_5, h_7, h_{11}, h_{13}$, where
   $$h_i(x) = \begin{cases} +1 & \text{if } i \text{ divides } x \\ -1 & \text{otherwise} \end{cases} .$$

2. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hypothesis classes with $\mathrm{VC}(\mathcal{H}_1) = d_1$ and $\mathrm{VC}(\mathcal{H}_2) = d_2$.

   (a) Prove an upper bound on $\mathrm{VC}(\mathcal{H}_1 \cup \mathcal{H}_2)$ (hint: use the fact that the growth function is less than $(em/d)^d$).

   (b) Prove an upper bound on $\mathrm{VC}(\mathcal{H}_1 \cdot \mathcal{H}_2)$, where $\mathcal{H}_1 \cdot \mathcal{H}_2$ is the class that includes hypotheses of the form $h(x) = h_1(x)h_2(x)$ for all $h_1 \in \mathcal{H}_1, h_2 \in \mathcal{H}_2$.