Lecture 1

Single-Item Mechanism Design

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Consider the following single-item mechanism design problem: There are n agents with true type $t_1, \ldots, t_n \in \mathbb{R}$, respectively. For each agent i, let b_i be the submitted bid of i. When receiving bids vector $\vec{b}$, the mechanism decides the allocation vector $A(\vec{b}) = \{q_1, \ldots, q_n\}$ and price vector $P(\vec{b}) = \{p_1, \ldots, p_n\}$. The $utility$ of each agent is

$$u_i(\vec{b}) = t_i \cdot q_i(\vec{b}) - p_i(\vec{b}).$$

We say a mechanism is $truthful$ (or *incentive compatible*) is for any t_i and b_{-i}, $u_i(\vec{b})$ is maximized when $b_i = t_i$.

According to the definition of truthfulness, we have

$$\frac{d}{db_i}\left[t_i \cdot q_i(\vec{b}) - p_i(\vec{b})\right]_{b_i=t_i} = 0.$$

Thus,

$$\left[t_i \cdot \frac{dq_i(\vec{b})}{db_i} - \frac{dp_i(\vec{b})}{db_i}\right]_{b_i=t_i} = 0,$$

which means

$$t_i q_i'(t_i) - p_i'(t_i) = 0.$$

Therefore,

$$\int_0^{b_i} p_i'(t_i)dt = \int_0^{b_i'} t_i q_i'(t_i)dt_i.$$

Hence,

$$p_i(b_i) = p_i(0) + [t_i \cdot q_i(t_i)]_{t_i=b_i} - \int_0^{b_i} q_i(t_i)dt_i = q_i(0) + b_i q_i(b_i) - \int_0^{b_i} q_i(t_i)dt_i. \quad (1.1)$$

The above formula and the condition of q_i is nondecreasing give a sufficient and necessary condition for truthfulness.

A mechanism is called *Bayes-Nash incentive compatible*, if for any t_i,

$$\int u_i(t_i, t_{-i}|t_i) f(t_{-i}) dt_i \geq \int u_i(b_i, t_{-i}|t_i) f(t_{-i}) dt_{-i},$$

where $f(t_{-i})$ is the probability distribution of t_{-i}.

Theorem 1.1. *(Revenue Equivalent Theorem) It's a Bayes-Nash incentive compatible if and only if Formula 1.1 holds and q_i is nondecreasing.*

$$
\begin{aligned}
\text{The expected revenue} &= \sum_i \int_{\vec{b}} p_i(\vec{b}) f(\vec{b}) d\vec{b} \\
&= \sum_i \int_{b_{-i}} \left[\int_{b_i} b_i q_i(b_i) f(b_i) db_i - \int_{b_i} \left[\int_{t=0}^{t=b_i} q_i(t) dt \right] f(b_i) db_i \right] f(b_{-i}) db_{-i} \\
&= \sum_i \int_{b_{-i}} \int_{b_i} \left[b_i - \frac{1 - f_i(b_i)}{f_i(b_i)} \right] q_i(b_i) f_i(b_i) db_i db_{-i} \\
&= \sum_i \int_b v_i(b_i) q_i(\vec{b}) f(\vec{b}) d\vec{b}, \quad \text{where } v_i(\vec{b}) \triangleq b_i - \frac{1 - f_i(b_i)}{f_i(b_i)} \\
&= \int_b \sum_i v_i(b_i) q_i(\vec{b}) f(\vec{b}) d\vec{b} \\
&= \int_b \left[\sum_A q_A(\vec{b}) \sum_{i \in A} v_i(b_i) \right] f(\vec{b}) d\vec{b}
\end{aligned}
$$

where A is any allocation and $q_A(\vec{b})$ is the probability that allocation A is selected as input $\vec{b}$.