Lecture 14

Cost Sharing Schemes

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Consider a scenario where a service is provided to a set of users at a certain net cost. A natural question to ask is, how should one divide the cost amongst the users, i.e., what is a good pricing policy or a *cost sharing scheme*? Formally, let U be the set of users and C be the cost function that assigns a service cost $C(S)$ to each $S \subseteq U$. It may help to keep the example of an internet service provider (ISP) in mind. In this case, U is the set of users that desire internet service and $C(S)$ is the monetary cost incurred by the ISP in providing the service to users in S. We are interested in a scheme that determines how this cost is divided up and recovered from the users.

Definition 14.1. A *cost sharing scheme* $\xi(i, S)$ assigns a cost to user $i \in S$ when the set of users S is served, i.e., $\xi : U \times 2^U \to \mathbb{R}^+$.

14.1 Properties of Cost Sharing Schemes

One can talk about several desired properties of a cost sharing scheme such as the following.

Budget-balanced: $\sum_{i \in S} \xi(i, S) = C(S)$. This says that the cost recovered from the users exactly matches the cost of providing the service.

Cross-monotonic: For all $i \in T \subset S$, $\xi(i, T) \geq \xi(i, S)$. This means that it never hurts a given user if more people join the service.

Core: For all $T \subset S$, $\sum_{i \in T} \xi(i, S) \leq C(T)$. This conditions implies that no subset of users has an incentive to break away from a bigger set of users that are currently being serviced. Note that any budget-balanced and cross-monotonic scheme is also cost sharing in the core.

Suppose we restrict ourselves to budget-balanced schemes. Does every cost function have a cross-monotonic cost sharing scheme? As the following example shows, the answer to this question is no. Let $U = \{1, 2, 3\}$, $C(U) = 2$, and for every $S \subset U, C(S) = 1$. Suppose ξ is a cross-monotonic cost sharing scheme for this scenario. By cross-monotonicity, $\xi(1, U) \leq \xi(1, \{1, 2\})$ and $\xi(2, U) \leq \xi(2, \{1, 2\})$. Adding these two inequalities and using the budget-balance condition, $\xi(1, U) + \xi(2, U) \leq \xi(1, \{1, 2\}) + \xi(2, \{1, 2\}) = C(\{1, 2\}) = 1$. Similarly, $\xi(2, U) + \xi(3, U)$ as well as $\xi(3, U) + \xi(1, U)$ are also each at most 1. Hence,

$\sum_{i \in U} \xi(i,U) = \frac{1}{2} \sum_{i \neq j \in U} (\xi(i,U) + \xi(j,U)) \leq 3/2$. This is strictly less than $C(U) = 2$, violating budget-balance. Therefore, no such cross-monotonic budget-balanced ξ exists.

This motivates a relaxation of the budget-balance condition that recovers only a fraction of the cost of providing the service.

Budget-balance with factor $\alpha < 1$: $\alpha C(S) \leq \sum_{i \in S} \xi(i,S) \leq C(S)$. This allows the charged cost to be as low as an α fraction of the cost of service.

Remark. The definition we use above is the one that occurs in literature more frequently. One could alternatively define a cost sharing scheme to be budget-balanced (in the relaxed sense) if $C(S) \leq \sum_{i \in S} \xi(i,S) \leq C(S)/\alpha$, allowing the service provider to charge a bit *more* than the cost of service. With a few simple relaxations elsewhere, this alternative definition can also be used to obtain the results that we will discuss.

14.2 Covering Games

We will use covering games to explore the connection between the budget-balance factor α and the integrality graph in linear programs formulations of several combinatorial problems. In particular, we will show that for the edge cover and facility location games, there is no (approximately) budget-balanced cross-monotonic schemes that can recover more than $\frac{1}{2}$ and $\frac{1}{3}$ of the service cost, respectively.

14.2.1 Set Cover Game

We begin with a relatively simple set cover game. Here we have elements from a universe U and a collection $\mathcal{A}$ of subsets of U. Each $A \in \mathcal{A}$ has an associated cost c_A. The cost of a collection of sets is the sum of the costs of the sets in it. Given $S \subseteq U$, the task is to find a minimum cost sub-collection of $\mathcal{A}$ such that every element in S is in some set in the sub-collection. We define this problem as a $\{0,1\}$ *integer program* over variables x_A that indicate whether or not set A is chosen to be in the sub-collection.

$$C(S) = \min \sum_{A \in \mathcal{A}} c_A x_A$$
$$\text{s.t. } \forall i \in S. \sum_{A \ni i} x_A \geq 1$$
$$x_A \in \{0,1\}$$

We will show that any feasible solution to the dual of the linear program (LP) relaxation of this integer program is cost sharing in the core. The LP relaxation and its dual are given below. A dual variable $y_{i,S}$ is used for the i^{th} inequality constraint for S in the primal LP.

Primal

$$\min \sum_{A \in \mathcal{A}} c_A x_A$$

$$\text{s.t. } \forall i \in S. \sum_{A \ni i} x_A \geq 1$$

$$x_A \geq 0$$

Dual

$$\max \sum_{i \in S} y_{i,S}$$

$$\text{s.t. } \forall A \in \mathcal{A}. \sum_{i \in A} y_{i,S} \leq c_A$$

$$y_{i,S} \geq 0$$

Claim 14.1. $\{\{\boldsymbol{y_S}\} : \boldsymbol{y_S}$ is a feasible dual solution$\} = \{\xi : \xi$ is in the core$\}$.

Proof. Suppose first that ξ is in the core. This implies that $\sum_{i \in A} \xi(i, S) \leq C(A) \leq c_A$, which proves that $y_{i,S} = \xi(i, S)$ is a feasible dual solution. Here the first inequality follows from the fact that ξ is in the core and the second one is true because all elements of A can be covered by simply picking the set A at cost c_A.

For the other direction, assume that $y_{i,S}$ is a feasible solution to the dual LP. We will show that $\sum_{i \in T} y_{i,S} \leq C(T)$, where $C(T)$ is the cost of covering all elements of T which corresponds to $\sum_{i=1}^{k} c_{A_i}$. Notice that $\sum_{i \in T} y_{i,S}$ is at most $\sum_{j=1}^{k} \sum_{i \in A_j} y_{i,S}$, which in turn, by the dual constraint, is at most $\sum_{j=1}^{k} c_{A_j} = C(T)$. This finishes the proof. □

It follows from the claim that the integrality gap of this LP is equal to the budget-balance factor α of cost sharing in the core (not necessarily cross-monotonic). Recall that budget-balance and cross-monotonicity together imply the core. We conclude that for budget-balance and cross-monotonic cost sharing schemes for set cover, the budget-balance factor is at most the integrality gap of the above LP. It is natural to ask whether we can do better, i.e., can we find better bounds on the budget-balance factor when considering only cross-monotonic cost sharing schemes? This is answered in the affirmative in the next section.

14.2.2 Edge Cover Game

The edge cover game can be thought of as the restriction of the set cover game where every set is of size 2. Given an undirected graph G and a subset S of its vertices, the edge cover problem is to minimize the number of edges of G chosen such that every vertex in S has at least one edge incident on it included in the chosen edges. For this game, the users U are the vertices of G and the cost function C is defined as $C(S) = \min_{\{F \subseteq E(G),\ F \text{ covers } S\}} |F|$.

For every set S, one can obtain a minimum edge cover of S by taking a maximum matching on S and adding one edge for every vertex that is not covered by the maximum matching. Using this fact, we can give a cost sharing scheme in the core with a budget-balance factor of $\frac{2}{3}$. We will now argue that if we require the scheme to be cross-monotonic as well, we cannot do better than a factor of $\frac{1}{2}$.

Theorem 14.1. *For every $\epsilon > 0$, there is no $(\frac{1}{2} + \epsilon)$-budget-balanced cross-monotonic cost sharing scheme for the edge cover problem.*

The proof of this theorem and the next will be based on the following **general scheme**.

1. Choose an instance G of the game.
2. Define a structure S over the vertices of G with some symmetry properties.

Graph $K_{n,n}$

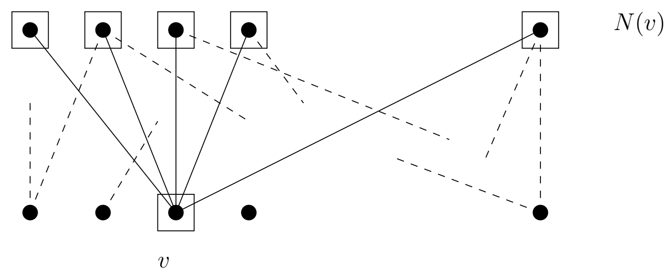

Figure 14.1: Graph $K_{n,n}$ for the edge cover game. S corresponding to vertex v chosen at random consists of the vertices marked with a square.

3. Use probabilistic method -
   - Choose S randomly and show that the expected value of $\sum_{i \in S} \xi(i, S)$ is at most $\alpha C(S)$.
   - Conclude that there exists an S such that $\sum_{i \in S} \xi(i, S) \leq \alpha C(S)$.

Proof of Theorem 14.1. Let $G = K_{n,n}$ be the complete bipartite graph with n vertices in each partition. For each vertex v in the second partition, define the structure S to contain v and its n neighbors $N(v)$ in the first partition (see Figure 14.1). For a random choice of v, this gives a random structure S. Note that for any v, the cost of covering S is $C(S) = n$ because we need to include all edges within S to cover all vertices in $N(v)$. We will show that over the choices of v, the expected value of recovered cost, $\sum_{i \in S} \xi(i, S)$, is at most $1 + \frac{n}{2}$, which will give the desired result.

$$\begin{aligned}
\mathbf{E}_v\left[\sum_{i \in S} \xi(i, S)\right] &= \mathbf{E}_v\left[\xi(v, S)\right] + \mathbf{E}_v\left[\sum_{i \in N(v)} \xi(i, S)\right] \\
&\leq 1 + \mathbf{E}_v\left[\sum_{i \in N(v)} \xi(i, \{i, v\})\right] \\
&= 1 + n\mathbf{E}_{i,v}\left[\xi(i, \{i, v\})\right] \\
&\leq 1 + \frac{n}{2}
\end{aligned}$$

Here the first inequality follows from the cross-monotonicity of ξ and the subsequent equality from the symmetry in the structure S. The last inequality is true because by budget balance, the sum of the cost shares of i and v in order to cover $\{i, v\}$ is at most one (can choose the single edge $(i, v) \in E(G)$) and averaged over all v, the expected share of each is at most $\frac{1}{2}$. $\square$

14.2.3 Metric Facility Location Game

The metric facility location problem is the following. Given a set of cities, a set of facilities with opening costs, and metric connection costs between cities and facilities, the task is to open a subset of facilities and connect each city to an open facility so as to minimize the total cost. In the corresponding game, the cities form the set of users U and the cost $C(S)$ of a subset S of cities is the cost of the minimum facility location solution for that subset.

Theorem 14.2. *Any cross-monotonic cost sharing scheme for the facility location game is at most $\frac{1}{3}$-budget-balanced.*

Proof. We will work with the following instance of the facility location problem. There are k sets $A_1, A_2, \ldots, A_k$ of m cities each, where $m = \omega(k)$ and $k = \omega(1)$. For every subset B of cities containing exactly one city from each A_i, there is a facility f_B with connection cost 1 to each city in B. The remaining connection costs are defined by extending the metric, i.e., the cost of connecting city $i \notin B$ to facility f_B is 3. The facility opening costs are all 3 (see Figure 14.2).

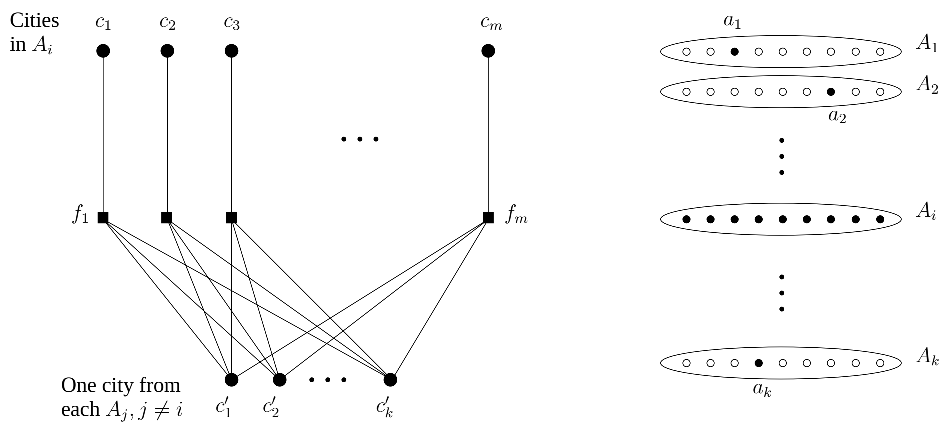

Figure 14.2: The figure on the left shows the facilities connected well to cities in A_i. The one on the right shows the structure of S corresponding to i; the filled nodes are included in S and the empty ones are not.

The random set S of cities is picked as follows. Pick a random $i \in \{1, 2, \ldots, k\}$ and for every $j \neq i$, pick a city a_j at random from A_j. Let $T = \{a_j : j \neq i\}$ and $S = A_i \cup T$ (see Figure 14.2). Note that $|T| = k - 1$ and $|A_i| = m$.

For any such S, the optimal cost of serving S is $3 + (k-1) + 1 + 3(m-1) = 3m + o(m)$, which can be achieved by opening one facility f with connection cost 1 to each city in T and cost 1 to a particular city in A_j, and connecting the remaining $m - 1$ cities in A_j to f at cost 3 each. We will show that the average cost recovered over choices of S is only $m + o(m)$, proving the result.

The expected cost recovered is given by

$$\mathbf{E}_S\left[\sum_{c \in S} \xi(c, S)\right] = \mathbf{E}_S\left[\sum_{c \in A_i} \xi(c, S)\right] + \mathbf{E}_S\left[\sum_{j \neq i} \xi(a_j, S)\right]$$

The second term on the right hand side of this equation is at most $\mathbf{E}_S\big[\sum_{j \neq i} \xi(a_j, T)\big]$ by cross-monotonicity, which is bounded above by $3 + (k-1) = k + 2$ by construction of the instance. The first term is at most $\mathbf{E}_S\big[\sum_{j \in A_i} \xi(j, T \cup \{j\})\big]$, again by cross-monotonicity. By symmetry, this is equal to $m\mathbf{E}_{j,S}\big[\xi(j, T \cup \{j\}\,\big]$. The random experiment chooses an A_i and $k-1$ random cities, one from each

$A_j, j \neq i$. This has the same distribution as the alternative random experiment that picks k random cities, one from each A_j, and then picks all cities from a randomly chosen A_i. Thinking of this alternative random experiment, the said expected value of $\xi(j, T \cup \{j\})$ is $\frac{1}{k}(k+3)$. This is because one can open a single facility connected at cost 1 to each of the k cities in $T \cup \{j\}$, and divide the cost equally among the cities. It follows that the whole term has an expected value of $\frac{m}{k}(k+3) = m + o(m)$. □

References

[1] N. Immorlica, M. Mahdian, M., and V.S. Mirrokni. Limitations of Cross-Monotonic Cost Sharing Schemes. In *Proceedings of the 16th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 602-611, Vancouver, BC, Jan 2005.