Lecture 1

Introduction to Cost Sharing

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1.1

Recall our definitions from the last lecture. Given a graph G and source-destination pairs (s_i, t_i), the volume of traffic from s_i to t_i is given by r_i. The latency on each edge is denoted by l_e and it's assumed to be continuous and non-decreasing.

The flow on a path p from s_i to t_i is given by f_p^i for all paths from s_i to t_i, P_i. The congestion on edge e is

$$g_e = \sum_{p \in \bigcup P_i, e \in p} f_p^i$$

The delay of a path p is $l_p = \sum_{e \in p} l_e(f_e)$.

We are at (Wardrop) equilibrium or Nash flow iff

$$\forall i \forall p' \forall p \in P_i \; s.t. \; f_p > 0, \; l_{p'} \geq l_p$$

$$\text{Price of Anarchy } = \frac{\text{social cost of worst equilibrium}}{\text{optimal social cost}}$$

In order to minimize the price of anarchy, we try to put "taxes" on edges. Let f_p flow such that $\sum_{p \in P_i} f_p = r_i$.

Question: Given a congestion goal $\overrightarrow{g}$, can we find tolls τ_e such that the congestion induced by a Nash flow is $\overrightarrow{g}$? Then $\overrightarrow{g}$ is enforced by $\overrightarrow{\tau}$.

Let's consider the case where each agent for traffic from s_i to t_i wishes to minimize $\alpha_i(\text{time spent}) + (\text{tolls paid})$. We introduce the following LP with the associated dual variables.

$$
\begin{aligned}
min \qquad & \sum_i \alpha_i \sum_{p \in P_i} l_p(g) f_p^i \\
dual : t_e \qquad & \forall e \in E, \sum_i \sum_{p \in P_i | e \in p} f_p^i \leq g_e \\
dual : z_i \qquad & \forall i, \sum_{p \in P_i} f_p^i = r_i \\
& f_p^i \geq 0
\end{aligned}
$$

The corresponding Dual program is

$$
\begin{aligned}
max \qquad & \sum_i r_i z_i - \sum_e g_e t_e \\
\forall i \forall p \in P_i \qquad & z_i - \sum_{e \in p} t_e \leq \alpha_i l_p(g) \\
\forall e \qquad & t_e \geq 0
\end{aligned}
$$

Definition 1.1. A congestion is *minimal* if the primal LP has an optimal solution in which all inequalities are tight.

Theorem 1.1. *A congestion g is enforceable by tolls iff g is minimal.*

Proof. To prove the first direction, assume that g is minimal, therefore exists an optimal f where all the constraints are tight. Let $(\overrightarrow{t}, \overrightarrow{z})$ be the optimal dual solution.

Fix i and by complementary slackness we get

- if $f_p^i > 0 \Rightarrow z_i = \sum_{e \in p} t_e + \alpha_i l_p(g)$
- if $f_p^i = 0 \Rightarrow z_i \leq \sum_{e \in p} t_e + \alpha_i l_p(g)$

We see that when the tolls are set at t_e, the Nash flow on every positive path has the same cheapest value.

To prove the other direction, assume that g is enforcable by tolls. This means that there exists flow f and tolls T such that f is a Nash flow and the congestion induced is g.

Let $z_i = \sum_{e \in P} T_e + \alpha_i l_p(g)$ be the value for any path with $f_p^i > 0$. But this means that there exists a feasible dual solution that satisfies the complementary slackness conditions, therefore all primal constraints are tight and g is minimal. $\square$

1.2 Coalition games

In a coalition game of N players, we consider the value assigned to subsets of players

$$v : 2^N \rightarrow \mathcal{R} \geq 0$$

or $v(S)$ the value of subset S working together. Typical assumptions are $v(\emptyset) = 0, S \subseteq T \Rightarrow v(S) \leq v(T)$.

Definition 1.2. Solution concept: "core"

$$(x_1, \ldots, x_n) \text{ is in the core if } \sum_{1 \leq i \leq n} x_i = v(N)$$

$$\text{and } \forall S, \sum_{i \in S} x_i \geq v(S)$$

The core property guarantees that no subset will want to work on their own.

1.2.1 Shapley axioms

Definition 1.3. The *marginal value* for i with respect to S is

$$\Delta_i(S) = v(S \cup \{i\}) - v(S)$$

The Shapley axioms are

1. **Dummy axiom:** If $\Delta_i(S) = \alpha_i, \forall S$ s.t. $i \notin S$ then $x_i = \alpha_i$.
2. **Symmetry:** If $\Delta_i(S) = \Delta_j(S), \forall S$ s.t. $i, j \notin S$ then $x_i = x_j$.
3. **Linearity:** If $v(S) = v_1(S) + v_2(S)$, then $x_i(v) = x_i(v_1) + x_i(v_2)$.

The way to obtain the Shapley value is to order the elements of N according to a permutation Π. Then

$$\begin{aligned} x_i^\Pi &= v(S \cup \{i\}) - v(S) \\ S.V. &= E_\Pi(x_i^\Pi \end{aligned}$$

It all comes down to this interesting theorem

Theorem 1.2. *The Shapley Value is the only way to satisfy the Shapley axioms.*

1.3 Cost Sharing

We would like to share the cost of a jointly utilized facility in a fair manner. We start by defining $c(S)$, the cost to serve subset S which is the same as before for $v(S) = -c(S)$. We still have the same solution concepts.

An example is multicast where a root serves nodes in a tree network. In this example, the Shapley value splits equally the cost of an edge on all users downstream of that edge.

The problem we're interested is an extension in which the players have utility values u_i, the utility to user i to receive service.

The desired mechanism properties are

1. **Efficiency:** The set of users that receive service is $S = argmax_T[\sum_{i \in T} u_i - c(T)]$.

2. **Budget-balance:** $\sum_{i \in S} x_i = c(S)$

3. **Truthful:**

The standard restrictions are

1. **Non-Positive Transfers:** $x_i \geq 0$

2. **Voluntary Participation:** $x_i = 0$ if $i \notin S$

3. **Consumer Sovereignty:** $\forall i \exists u_i, i \in S$

Fact 1.1. It's not possible to get budget-balance and efficiency at the same time.

Let's focus on truthful budget-balanced mechanisms. Let $\xi(i, S)$ be the payment by player $i \in S$ if S is the set receiving service.

$$\sum_{i \in S} \xi(i, S) = c(S)$$

Players knowing ξ say yes/no and those that say yes (set S) receive service and pay $\xi(i, S)$. The mechanism should look like this

$$u_1, \ldots, u_n \to q_i(u_i) \to \Box \to S = \{i | q_i \text{ is yes}\}, \xi(i, S)$$

Revelation principle: If there is a unique dominant strategy equilibrium then there exists a truthful mechanism. Just have players reveal u_i's and compute q_i's in the mechanism.

We say that ξ is cross-monotonic if $\forall S \subseteq T \ i \in S \ \xi(i, T) \leq \xi(i, S)$.

Truthful mechanism: Ask for utilities and offer payments $\xi(i, N)$. Some are happy and say yes while others say no. Then let S be those that are happy. Offer them $\xi(i, S)$ and keep doing that.

Finally, we should mention the case of submodular cost functions

$$\forall A \subseteq B \ \ c(A \cup \{i\}) - c(A) \geq c(B \cup \{i\}) - c(B)$$

In this case, the Shapley value is a cross-monotonic cost sharing mechanism.