Lecture 1

Introduction to Game Theory

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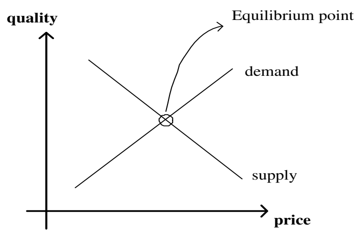
Demand and Supply curve for one goods:



In the following we consider how to generalize this definition to n goods.

1.1 Fisher Setting

There are n buyers, denoted by $i = 1, \ldots, n$, with initial money $m_1, \ldots, m_n$, respectively. There are q heterogeneous goods, denoted by $j = 1, \ldots, q$. Let B_j denote the amount of goods j. For each buyer i, let u_{ij} be the utility of buyer i from goods j. Let u_i be a *utility function* defined on $\mathbb{R}^q \to \mathbb{R}$, representing the utility i gets from the allocation vector.

Given price vector $p = (p_1, \ldots, p_q)$, each buyer tries to maximize its own utility according to the following program:

$$\begin{array}{rl} \max & u_i(g) \\ s.t. & p \cdot g \leq m_i \\ & g \geq 0, \text{ where } g \in R^q \end{array}$$

Assume the utility function of each buyer is $concave$, that is,

$$\frac{u_i(g_1) + u_i(g_2)}{2} \leq u_i\left(\frac{g_1 + g_2}{2}\right).$$

Then the above program can be solved in polynomial time by ellipsoid algorithm. For each goods j, we assume there is at least one i such that $u_{ij} > 0$.

Similar to the Demand and Supply curve for one goods, we can see there are the following three cases for each goods j:

Case 1. total demand $>$ supply, p_j should increase.

Case 2. total demand $<$ supply, p_j should decrease.

Case 3. total demand $=$ supply, p_j should not change.

Theorem 1.1. *(Arrow & Debreu, 1954) There exist price vector $p = (p_1, \ldots, p_q)$ and allocation vector $g = (g_1, \ldots, g_n)$ such that (i) w.r.t p, the utility of each buyer is maximized, and (ii) for each goods j, $\sum_i g_{ij} = B_j$.*

Any solution satisfies the above two conditions is called *market equilibrium*.

1.2 Linear Utility Functions

In the following we consider linear utility functions, that is, for each buyer i,

$$u_i = \sum_j u_{ij} \cdot g_{ij},$$

where g_{ij} is the amount of goods j that i gets.

$$\begin{aligned}
\max \qquad & \prod_{i=1}^{n} u_i^{m_i} \\
s.t. \qquad & \sum_i x_{ij} \le B_j = 1, \forall\, j \\
& x_{ij} \ge 0, \forall\, i, j
\end{aligned}$$

where x_{ij} is the allocation of goods j to buyer i, and assume w.l.o.g that $B_j = 1$ for each j.

The objective function is equivalent to

$$\max \sum_{i=1}^{n} m_i \cdot \log \left(\sum_j u_{ij} \cdot x_{ij} \right).$$

Due to Lagrange relaxation, we have for any fixed price vector p,

$$\begin{aligned}
\max \qquad & \sum_{i=1}^{n} m_i \cdot \log \left(\sum_j u_{ij} \cdot x_{ij} \right) + \sum_j p_j \cdot (1 - \sum_i x_{ij}) \ge \sum_{i=1}^{n} m_i \cdot \log \left(\sum_j u_{ij} \cdot x_{ij} \right) \\
& x_{ij} \ge 0, \forall\, i, j
\end{aligned}$$

If $x_{ij} > 0$,

$$\frac{m_i \cdot u_{ij}}{\sum_j u_{ij} \cdot x_{ij}} - p_j = 0 \;\Rightarrow\; \frac{u_{ij}}{p_j} = \frac{\sum_j u_{ij} \cdot x_{ij}}{m_i}.$$

If $x_{ij} = 0$,

$$\frac{m_i \cdot u_{ij}}{\sum_j u_{ij} \cdot x_{ij}} - p_j \leq 0 \;\Rightarrow\; \frac{u_{ij}}{p_j} \leq \frac{\sum_j u_{ij} \cdot x_{ij}}{m_i}.$$

Thus, we know each buyer spends its money with utility maximized. In addition, we know $1 - \sum_i x_{ij} = 0$, which means all goods are clear. For any i, j (no matter $x_{ij} = 0$ or $x_{ij} \neq 0$), we have

$$u_{ij}x_{ij} = \frac{(\sum_j u_{ij}x_{ij})x_{ij}p_j}{m_i},$$

which implies that

$$\sum_j u_{ij}x_{ij} = \sum_j \frac{(\sum_j u_{ij}x_{ij})x_{ij}p_j}{m_i}.$$

Thus,

$$1 = \frac{\sum_j p_j x_{ij}}{m_i},$$

which implies all buyers spend their money. Therefore, the solution of the above program gives a market equilibrium.

Basically, these are what we have discussed in class. Please refer to the reference for more details. (More details to be included).

References

[1] K. Jain, *A Polynomial Time Algorithm for Computing the Arrow-Debreu Market Equilibrium for Linear Utilities*, FOCS 2004, 286-294.