# Balanced Metric Labeling

## [Extended Abstract]

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## ABSTRACT

We define the *balanced metric labeling* problem, a generalization of the metric labeling problem, in which each label has a *capacity*, i.e., at most $\ell$ vertices can be assigned to it. The balanced metric labeling problem is a generalization of fundamental problems in the area of approximation algorithms, e.g., arrangements and balanced partitions of graphs. It is also motivated by resource limitations in certain practical scenarios. We focus on the case where the given metric is uniform and note that this case alone encompasses various well-known graph partitioning problems. We present the first (pseudo) approximation algorithm for this problem, achieving for any $\varepsilon$, $0 < \varepsilon < 1$, an approximation factor of $O\left(\frac{\ln n}{\varepsilon}\right)$, while assigning at most $\min\left\{\frac{O(\ln k)}{1-\varepsilon}, \ell+1\right\}(1+\varepsilon)\,\ell$ vertices to each label ($k$ is the number of labels). Our approximation algorithm is based on a novel randomized rounding of a linear programming formulation that combines an embedding of the graph in a simplex together with spreading metrics and additional constraints that strengthen the formulation. Our randomized rounding technique uses both a randomized metric decomposition technique and a randomized label assignment technique. At the heart of our approach is the fact that only limited dependency is created between the labels assigned to different vertices, allowing us to bound the expected cost of the solution and the number of vertices assigned to each label, simultaneously. We note that the number of vertices assigned to each label is bounded via a new inequality of Janson [15] for tail bounds of (partly) dependent random variables.

## Categories and Subject Descriptors

G.2.2 [**Discrete Mathematics**]: Graph Theory—*Graph Labeling, Graph Algorithms*

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## General Terms

Algorithms, Theory

## Keywords

Metric-Labeling, Balanced-Partition

## 1. INTRODUCTION

The *metric labeling* problem captures a wide variety of classification problems that arise in computer vision and other related areas. Typically, in a classification problem one wishes to assign labels to objects such that some measure of the quality of the labeling is optimized. Metric labeling captures many classification problems where the measure of the quality of the labeling depends not only on the cost of assigning a label to an object, but also on pairwise relations between the classified objects. Formally, the input for the metric labeling problem consists of an undirected graph $G = (V, E)$ with $n$ vertices, a non-negative edge weight function $w : E \to \mathbb{R}^+$, a set $L = \{1, 2, \ldots, k\}$ of $k$ labels, a metric $(L, d_L)$ over the set of labels and a non-negative assignment cost function $c : V \times L \to \mathbb{R}^+$. The goal is to find a labeling of the vertices $f : V \to L$ that minimizes: $\sum_{u \in V} c\big(u, f(u)\big) + \sum_{(u,v) \in E} w(u,v) \cdot d_L\big(f(u), f(v)\big)$.

Kleinberg and Tardos [16] were the first to consider the metric labeling problem in its full generality. They presented a 2-approximation for the uniform metric case and an approximation of $O(\log k \log\log k)$ for general metrics using the probabilistic tree embedding technique [5, 4]. This bound was recently improved to $O(\log k)$ [12]. We note that several special cases of the metric labeling problem received attention, e.g., [14, 6, 2]. Chuzhoy and Naor [7] showed that metric labeling is $\Omega(\log^{\frac{1}{2}-\delta} k)$-hard to approximate under the assumption that NP $\not\subseteq$ DTIME $\left(n^{\mathrm{poly}(\log n)}\right)$. Chekuri *et al.* [6] formulated a linear programming relaxation for metric labeling, where each vertex is associated with a distribution over the label set $L$, i.e., an embedding in a $k$-dimensional simplex, and distances are defined by an *earth-mover* metric.

Metric labeling also generalizes some classical and extensively studied combinatorial optimization problems. In the *0-extension* problem, there is a special set of terminals $\{t_1, t_2, \ldots, t_k\}$, where each terminal $t_i$ is a priori assigned label $i$. There is a metric on the label set, however, the assignment cost of labels to non-terminal vertices is zero. An $O(\log k)$-approximation algorithm for the problem was given by [8] and later improved to an $O(\log k / \log\log k)$-

approximation [11]. The special case of a uniform metric is the well known *multiway cut* problem.

## 1.1 Balanced Metric Labeling

We introduce the *balanced metric labeling* problem in this paper. In this problem, each label has a *capacity*, $\ell$, which is an upper bound on the number of vertices that can be assigned to it. The balanced metric labeling problem is a natural generalization of several well known graph partitioning and arrangement problems. For example, the $b$-balanced cut problem, $0 < b \leq 1/2$, asks for a minimum capacity cut $(S, V \setminus S)$ such that $b \cdot n \leq |S| \leq (1-b) \cdot n$. Alternatively, we can say that each vertex can be assigned to one of two labels (at zero cost), indicating the side of the balanced cut to which the vertex belongs. Setting $\ell = (1-b)n$, the resulting labeling is indeed a $b$-balanced cut. The metric $d_L$ in this case is the uniform metric, and the cost of the labeling is the total weight of edges whose endpoints are in different sides of the cut. Similarly, the $k$-way balanced partitioning problem, where the goal is to partition a graph into $k$ (almost) equal parts, is also a special case of the balanced metric labeling problem. Here, $|L| = k$ and the metric is uniform.

In the linear arrangement problem, the goal is to embed a graph onto the integral points of a straight line, while minimizing the sum of the lengths of the edges in the embedding. This is a special case of the balanced metric labeling, by setting $L = \{1, 2, \ldots, n\}$, $\ell = 1$, and letting $d_L$ be a linear metric. Thus, the resulting labeling defines a linear arrangement of the graph, and the cost of the labeling is the total length of the edges in the embedding. It is important to notice that *any* embedding of an undirected graph onto a finite metric space, where the goal is to minimize the volume of the embedding (total edge length in the embedding), is a special case of the balanced metric labeling problem. Thus, this problem encompasses various types of linear arrangements and embeddings of graphs onto grids and meshes, in any dimension.

An interesting application of our problem arises in the design of wireless networks. Consider, for example, a GSM network. A Base Transceiver Station (BTS) provides coverage to mobile users in a cell. The BTS-s are further partitioned into clusters, where each cluster is controlled by a Base Station Controller (BSC). Handover, the maintenance of service to users moving between cells, is much simpler (or cheaper) within a cluster. We assume that each BTS can only be assigned to a subset of the BSC-s, and a weighted graph is defined on the BTS-s, where edge weights correspond to the traffic between pairs of BTS-s. Thus, viewing the BSC-s as labels, where each BSC can control only a limited number of BTS-s, finding a minimum cost clustering of the BTS-s is an instance of the balanced metric labeling problem with a uniform metric and labeling costs belonging to $\{0, \infty\}$.

## 1.2 Our Results

We consider the balanced metric labeling problem with a uniform metric. As already mentioned, this case by itself is already a generalization of many partitioning problems. We present the first (pseudo) approximation for this problem. For any $0 < \varepsilon < 1$, we find an $O\left(\frac{\ln n}{\varepsilon}\right)$-approximation, such that there are at most $\min\left\{\frac{O(\ln k)}{1-\varepsilon}, \ell+1\right\}(1+\varepsilon)\,\ell$ vertices assigned to each label. We note that for $\ell = O(1)$ or $k = O(1)$ (which is the case in many partitioning problems), we obtain a constant multiplicative deviation in the number of vertices that receive the same label. Compare our result with the best known results for balanced graph partitioning: a pseudo-approximation was given by [17, 9], where the cost is approximated by an $O(\log n)$ factor, and the multiplicative deviation in the number of vertices assigned to a part ("label") is a constant. Very recently, the $O(\log n)$ factor was improved to $O(\sqrt{\log n})$ [3]. A (true) $O(\log^2 n)$-approximation algorithm for 2-way balanced cuts was given by [13]. However, it does not seem that any of the above results can be naturally extended to handle assignment costs of vertices to different parts ("labels").

There are two main difficulties that we need to cope with in order to approximate the balanced uniform metric labeling problem. First, there is no obvious way how to bound the number of vertices assigned to a label in the approximation algorithms developed for the (uncapacitated) uniform metric labeling problem, e.g. [16]. Second, it is not clear how to incorporate label assignment costs with techniques developed for approximating partitioning problems, e.g., spreading metrics [9] and the technique of [3]. Metric decomposition techniques yield a partitioning of the metric into clusters with small radii, such that if the metric satisfies some spreading constraints, this usually implies that each cluster contains a small number of vertices. However, in general, there may not always exist a label that can be assigned to all vertices in a single cluster of the partition, for example, if some labels cannot be assigned to certain vertices (by setting the assignment cost to be $\infty$).

We formulate the balanced uniform metric labeling problem as a linear program combining the simplex embedding approach [6, 16] together with the spreading metric approach [10, 9]. The simplex embedding approach can be used to bound the assignment and edge costs, while the spreading metric approach is useful for bounding the number of vertices assigned to the same label. However, the combination of these two approaches is not enough and we strengthen the relaxation by adding a new constraint which can be viewed as a triangle inequality on the "closeness" of vertices (see Section 3 and constraint (6) in the LP relaxation).

Our approximation algorithm combines in a novel way randomized metric decomposition techniques and randomized label assignment techniques. This needs to be done in a way that balances the dependencies between the labels assigned to different vertices. If each vertex is assigned a random label independently, then it is easy to bound the number of vertices that are assigned to the same label, however, the expected cost of the labeling in this case can be unbounded (compared with the LP). If the labels chosen for the vertices are completely dependent (e.g., [16]), then it is easy to bound the cost of the solution, however, there is no way to bound the number of vertices that are assigned to the same label. Additionally, one can find examples where the algorithm of [16], in which all labels are dependent, assigns all vertices to the same label, even when adding capacity constraints to the relaxation of [16]. Therefore, it is a challenging problem to achieve a poly-logarithmic approximation factor while deviating as little as possible from the capacities.

Hence, we present a novel approximation technique that creates a delicate balance of dependencies, i.e., the label

that a single vertex receives depends on only a limited number of other labels. This allows us to bound both the cost of the labeling and the number of vertices assigned to each label via a new inequality of Janson [15] for tail bounds of (partly) dependent random variables. We achieve a logarithmic approximation factor and a low (i.e. logarithmic in $k$) multiplicative deviation in the number of vertices assigned to each label. We note that the deviation is larger than the deviation achieved for partitioning problems [17, 9, 3], since we have to cope with assignment costs in the range $[0, \infty)$, instead of a fixed assignment cost of zero.

## 2. PRELIMINARIES

The input for the balanced uniform metric labeling problem consists of an undirected graph $G = (V, E)$ with $n$ vertices, a non-negative edge weight function $w : E \to \mathbb{R}^+$, a set $L = \{1, 2, \ldots, k\}$ of $k$ labels, a non-negative assignment cost function $c : V \times L \to \mathbb{R}^+$, and a capacity parameter $\ell$. The goal is to find a labeling of the vertices $f : V \to L$ that minimizes: $\sum_{u \in V} c\big(u, f(u)\big) + \sum_{(u,v) \in E} w(u, v) \cdot d_{unif}\big(f(u), f(v)\big)$, such that $\forall j \in L$, $|\{u \mid f(u) = j\}| \leq \ell$. We denote the uniform metric by $d_{unif}$, i.e., $\forall i, j \in L$, $i \neq j$, $d_{unif}(i, j) = 1$, and $\forall i \in L$, $d_{unif}(i, i) = 0$.

We note that we can determine in polynomial time using flow techniques whether a feasible solution having finite cost exists for a given instance. We can assume that $G$ is a complete graph by setting the weight of all missing edges to 0. We use the following density function (similarly to [5]).

DEFINITION 1. *$R \sim$ Radius($\varepsilon$), for $r \in \left[0, \frac{\varepsilon}{1+\varepsilon}\right]$, if its density is:*

$$f_R(r) = \left(\frac{n}{n-1}\right) \cdot \frac{1+\varepsilon}{\varepsilon} \ln n \cdot n^{-r \cdot \frac{1+\varepsilon}{\varepsilon}}.$$

Clearly, Radius($\varepsilon$) is a valid probability measure for any $\varepsilon > 0$.

LEMMA 1. *For any $0 \leq r_1 \leq r_2 \leq \frac{\varepsilon}{1+\varepsilon}$,*

$$\mathbb{P}r[0 \leq R \leq r_1] = \frac{n}{n-1} \cdot \left(1 - e^{-r_1 \frac{1+\varepsilon}{\varepsilon} \ln n}\right).$$

$$\mathbb{P}r[r_1 \leq R \leq r_2] = \frac{n}{n-1} \cdot \left(1 - e^{-(r_2 - r_1) \frac{1+\varepsilon}{\varepsilon} \ln n}\right) \cdot e^{-r_1 \frac{1+\varepsilon}{\varepsilon} \ln n}.$$

$$\mathbb{P}r\left[r_2 \leq R \leq \frac{\varepsilon}{1+\varepsilon}\right] = \frac{n}{n-1} \cdot \left(e^{-r_2 \frac{1+\varepsilon}{\varepsilon} \ln n} - \frac{1}{n}\right).$$

## 3. THE RELAXATION

We formulate the balanced uniform metric labeling problem as a linear program that combines spreading constraints with an embedding in a $k$-dimensional simplex, in which distances are bounded from below by the variation distance ($\ell_1$-distance), together with some additional new constraints.

In our formulation, each vertex is associated with a probability measure over the label set $L$, similarly to [6, 16]. Hence, vertex $v$ is associated with variables $\varphi(v, j)$, $1 \leq j \leq k$. The *probability measure* of $v$ over $L$ is a vector $\varphi(v) = \big(\varphi(v, 1), \varphi(v, 2), \ldots, \varphi(v, k)\big)$ (see constraints (1)). Furthermore, we add *capacity* constraints (see constraints (2)) which guarantee that each label is not assigned to more than $\ell$ vertices.

We introduce a set of variables, $c_j(u, v)$, $1 \leq j \leq k$, $u, v \in V$, where $c_j(u, v)$ measures the "closeness" of $\varphi(u)$ and $\varphi(v)$ with respect to label $j$ (see constraints (3)). Each edge $(u, v)$ is also associated with a distance $d(u, v)$ which is related to the probability measures via the closeness variables (see constraints (4)). The variation distance of the points $\varphi(v)$, $v \in V$, in the $k$-dimensional simplex is used to bound $d$ from *below*. It is important to notice that $d$ is a variable of the LP relaxation and not part of the input. Notice that constraints (1), (3) and (4) are equivalent to the relaxation of [16] for the uncapacitated uniform case.

The spreading constraints (see constraints (5)) imply that each sphere of bounded radius does not contain too many vertices. We strengthen the relaxation by adding a *new* constraint that can intuitively be thought of as a "triangle inequality" constraint for the closeness variables. We call these constraints *closeness triangle inequality* (see constraints (6)). These constraints are important for bounding the separation cost of neighboring vertices and we note that they can be made linear. Additionally, these constraints (with the variation distance constraints (4)) imply that $(V, d)$ is a semi-metric (see Lemma 2).

$$\min \quad \sum_{v \in V} \sum_{j \in L} c(v, j) \cdot \varphi(v, j) + \sum_{u, v \in V} w(u, v) \cdot d(u, v) \quad \text{s.t.}$$

$$\sum_{j \in L} \varphi(v, j) = 1$$
$$\forall v \in V \qquad \text{(probability measure)} \tag{1}$$
$$\sum_{v \in V} \varphi(v, j) \leq \ell$$
$$\forall j \in L \qquad \text{(capacity)} \tag{2}$$
$$c_j(u, v) \leq \varphi(u, j), \varphi(v, j)$$
$$\forall u, v \in V, \forall j \in L \qquad \text{(closeness)} \tag{3}$$
$$d(u, v) = 1 - \sum_{j \in L} c_j(u, v)$$
$$\forall u, v \in V \qquad \text{(variation distance)} \tag{4}$$
$$\sum_{v \in S} d(u, v) \geq |S| - \ell$$
$$\forall S \subseteq V, \forall u \in S \qquad \text{(spreading)} \tag{5}$$
$$\sum_{j \in L} \big|c_j(u, v) - c_j(u, w)\big| \leq 1 - \sum_{j \in L} c_j(v, w)$$
$$\forall u, v, w \in V \qquad \text{(closeness } \Delta\text{−inequal.)} \tag{6}$$
$$\varphi(v, j) \geq 0, c_j(u, v) \geq 0, d(u, v) \geq 0$$
$$\forall u, v \in V, \forall j \in L$$

LEMMA 2. *For any feasible solution for the above program, $(V, d)$ is a semi-metric.*

PROOF. For every $u, v \in V$, $d(u, v) \geq 0$. Thus, we only need to prove the triangle inequality. For every $u, v, w \in V$, according to constraint (6) (closeness $\Delta$-inequality):

$$\sum_{j \in L} \big|c_j(u, w) - c_j(u, v)\big| \leq 1 - \sum_{j \in L} c_j(w, v).$$

Removing the absolute value from the left hand side of the equation can only decrease its value, therefore:

$$\sum_{j \in L} \big(c_j(u, w) - c_j(u, v)\big) \leq 1 - \sum_{j \in L} c_j(w, v).$$

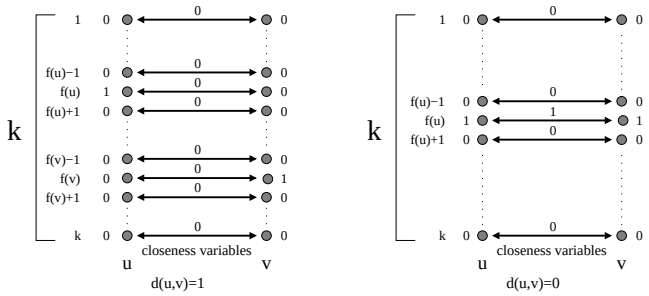



**Figure 1: Variable Assignments.**

Rearranging terms, we get that:

$$1 - \sum_{j \in L} c_j(u,v) \leq 1 - \sum_{j \in L} c_j(u,w) + 1 - \sum_{j \in L} c_j(w,v).$$

Applying constraint (4) (variation distance) yields the triangle inequality: $d(u,v) \leq d(u,w) + d(w,v)$, thus completing the proof. □

LEMMA 3. *The value of an optimal solution for the above program is a lower bound on the cost of an optimal solution to the balanced uniform metric labeling problem.*

PROOF. Given an instance of the balanced uniform metric labeling problem, we show that it defines a feasible solution to the linear program, such that the objective function value of the linear program is equal to the cost of the labeling $f$.

Let us define a solution as follows. For each vertex $v$, set $\varphi(v, f(v)) = 1$ and $\varphi(v,j) = 0$ for every $j \neq f(v)$. Next, set for every two vertices $u, v \in V$ and every $j \in L$, $c_j(u,v) = \min\{\varphi(u,j), \varphi(v,j)\}$, and $d(u,v) = 1 - \sum_{j \in L} c_j(u,v)$. Thus, if $f(u) = f(v)$, then $c_j(u,v) = 0$ for every $j \neq f(u)$ and $c_{f(u)}(u,v) = 1$. Otherwise, $f(u) \neq f(v)$ and $c_j(u,v) = 0$ for every $j \in L$. The assignments can be viewed in Figure 1.

Clearly, this solution satisfies constraints (3) (closeness constraints), (4) (variation distance constraints), and also constraints (1) (probability measure constraints). Since the labeling $f$ does not assign more than $\ell$ vertices to each label, constraints (2) (capacity constraints) are also satisfied. Note that $d(u,v) = 0$ if $f(u) = f(v)$, and $d(u,v) = 1$ otherwise. Let us now focus on constraints (5) (spreading constraints). Let $S \subseteq V$ and $u \in S$. Since $f$ assigns at most $\ell$ vertices to each label $j \in L$, in particular for label $f(u)$ we get that: $|\{x \in S \mid f(u) \neq f(x)\}| \geq |S| - \ell$. For every $v \in \{x \in S \mid f(u) \neq f(x)\}$, $d(u,v) = 1$, hence the solution satisfies constraints (5).

Let us now focus on constraints (6) (closeness $\Delta$-inequality constraints). If $v$ and $w$ are assigned the same label ($f(v) = f(w)$), then $d(v,w) = 0$ (or equivalently $\sum_{j \in L} c_j(v,w) = 1$) and $c_j(u,v) = c_j(u,w)$, $\forall j \in L$. Thus, the solution satisfies the constraint in this case. Otherwise, $v$ and $w$ are assigned different labels ($f(v) \neq f(w)$), therefore $d(v,w) = 1$ (or equivalently $\sum_{j \in L} c_j(v,w) = 0$). If $u$ is assigned to the same label as $v$ or $w$ ($f(u) \in \{f(v), f(w)\}$), the left hand side of the constraint is equal to 1 and the solution satisfies the constraint. If $u$ is not assigned the same label as $v$ and $w$ ($f(u) \notin \{f(v), f(w)\}$), the left hand side of the constraint equals 0, and the solution satisfies the constraint. Thus, we conclude that constraints (6) are satisfied, and hence the solution is feasible.

We now examine the value of the solution. Clearly, for each vertex $v \in V$, $\sum_{j \in L} c(v,j)\varphi(v,j) = c\big(v, f(v)\big)$. Since $d(u,v) = 0$ if $f(u) = f(v)$, and $d(u,v) = 1$ otherwise, we conclude that $\forall u, v \in V$,

$$w(u,v)d(u,v) = w(u,v)d_{unif}(f(u), f(v)).$$

Therefore, the value of the objective function is precisely the cost of the labeling, completing the proof. □

LEMMA 4. *An optimal solution for the above program is computable in polynomial time.*

From the proof of Lemma 2 it follows that the closeness $\Delta$-inequality constraints (constraint (6)), together with the variation distance constraints (constraint (4)), imply the triangle inequality for $d$. An interesting question is whether the converse is also true, i.e., can we prove that any feasible solution to the linear program that satisfies the triangle inequality constraints for $d$ together with all the other constraints, except for constraint (6), also satisfies the closeness $\Delta$-inequality.

The answer to this question is negative as we give an example in which the triangle inequality for $d$ is satisfied, yet the closeness $\Delta$-inequality is not satisfied. Consider a graph with only three vertices $u$, $v$ and $w$, two labels, and capacity $\ell = 2$. Set the following distributions on the labels: $\varphi(u) = \varphi(v) = \varphi(w) = (\frac{1}{2}, \frac{1}{2})$. Set the closeness variables in the following way: $c_1(u,v) = \frac{1}{2}$, $c_2(u,v) = 0$, $c_1(u,w) = 0$, $c_2(u,w) = \frac{1}{2}$, $c_1(v,w) = \frac{1}{4}$ and $c_2(v,w) = \frac{1}{4}$. It can be verified that all the constraints, except for the closeness $\Delta$-inequality, are satisfied. According to constraint (4) (variation distance), $d(u,v) = d(u,w) = d(v,w) = \frac{1}{2}$. Clearly, the triangle inequality for $d$ is satisfied. However, the closeness $\Delta$-inequality is not satisfied, since:

$$\sum_{j=1}^{2} \big|c_j(u,v) - c_j(u,w)\big| = 1 > \frac{1}{2} = 1 - \sum_{j=1}^{2} c_j(v,w) = d(v,w).$$

This implies that the closeness $\Delta$-inequality constraint is stronger than an explicit constraint stating that $d$ satisfies the triangle inequality.

# 4. THE APPROXIMATION ALGORITHM

Our approximation algorithm combines randomized metric decomposition techniques with randomized label assignment techniques. We prove that our technique creates limited dependencies between the labels that vertices are assigned to. This enables us to bound both the cost of the

solution and the number of vertices assigned to the same label, simultaneously.

Our approximation algorithm first assigns an initial labeling denoted by $f^*$, which we call a *root labeling*. This is done by a random procedure **RootLabel** and we defer its description to Section 5.2. The only property we need is that for every $v \in V$, $f^*(v)$ is a random variable satisfying:

$$\mathbb{P}\mathrm{r}[f^*(v) = j] = \varphi(v, j) \quad , \quad \forall v \in V, \forall j \in L. \tag{7}$$

We do not assume that the random variables $\{f^*(v)\}_{v \in V}$ generated by **RootLabel** are necessarily independent. However, we assume that the random variables $\{f^*(v)\}_{v \in V}$ are independent of any other random choices made during our algorithm. We note that Property (7) is required for bounding the expected cost of the solution found by our approximation algorithm.

Our approximation algorithm, **ApproxLabel**, assigns labels to vertices in iterations. **ApproxLabel** conducts a single iteration for each vertex, where the vertex that corresponds to an iteration is called the *root* vertex of that iteration. We assume that the root vertices are determined in each iteration according to a prespecified ordering of the vertices $V = \{u_1, u_2, \ldots, u_n\}$. This order is arbitrary and independent of any random choice made throughout the algorithm. In each iteration, the root label of that iteration's root vertex is assigned to a random (possibly empty) subset of vertices that are still unassigned to any label.

Let us consider iteration $i$ with root vertex $u_i$. The random subset chosen in such an iteration is constructed from the intersection of two other random subsets. The first random subset consists of all vertices that belong to a sphere of radius $R$ around $u_i$, where $R \sim Radius(\varepsilon)$. Hence, the first subset is obtained via a randomized metric decomposition technique. The second subset consists of all vertices that are "close" to $u_i$ with respect to the root label of $u_i$, i.e. $f^*(u_i)$. Vertex $v$ is said to be "close" to vertex $u_i$ with respect to label $f^*(u_i)$, if $c_{f^*(u_i)}(u_i, v) \geq \alpha$, where $\alpha \sim Unif[0, \varphi(u_i, f^*(u_i))]$. Recall that:

$$c_{f^*(u_i)}(u_i, v) \leq \min\{\varphi(u_i, f^*(u_i)), \varphi(v, f^*(u_i))\},$$

and the variable $c_{f^*(u_i)}(u_i, v)$ denotes the "closeness" of vertices $v$ and $u_i$ in terms of label $f^*(u_i)$. Hence, the second subset is obtained via a randomized label assignment technique. Thus, the random subset chosen in iteration $i$ is the following intersection of two random subsets:

$$\{x \mid d(u_i, x) \leq R\} \cap \left\{x \mid c_{f^*(u_i)}(u_i, x) \geq \alpha\right\}.$$

The subsets can be viewed in Figure 2. We assume that all random variables chosen by **ApproxLabel** are independent of each other. That is, for each iteration, the values of $R$ and $\alpha$'s are independent of each other, and are also independent of any other value of $R$ and $\alpha$ chosen in any other iteration. We are now ready to state the approximation algorithm. We assume that an optimal solution for the linear program of the relaxation has been computed. At the end of each iteration we denote by $V'$ the subset of vertices that are still not assigned to any label.

**Algorithm ApproxLabel**

1. $f^* \leftarrow$**RootLabel**$(\{\varphi(u)\}_{u \in V})$.
2. $V' \leftarrow V$.
3. For $i = 1$ to $n$ do:
4. $\quad R \sim Radius(\varepsilon)$ and $\alpha \sim Unif\,[0, \varphi(u_i, f^*(u_i))]$.
5. $\quad S_i \leftarrow \{x \mid d(u_i, x) \leq R\} \cap \left\{x \mid c_{f^*(u_i)}(u_i, x) \geq \alpha\right\}$.
6. $\quad$ For each $v \in V' \cap S_i$, set $f(v) \leftarrow f^*(u_i)$.
7. $\quad V' \leftarrow V' \setminus S_i$.
8. Output $f$.

LEMMA 5. *All vertices are labeled by* **ApproxLabel**.

PROOF. For every $v \in V$, $d(v, v) = 0$. Hence, $c_j(v, v) = \varphi(v, j)$ for every $j \in L$, due to constraints (3) (closeness constraints), (4) (variation distance constraints), and (1) (probability measure constraints). Thus, $u_i \in \{x \mid d(u_i, x) \leq R\} \cap \left\{x \mid c_{f^*(u_i)}(u_i, x) \geq \alpha\right\}$, for any possible value of $R$, $i$, $\alpha$, and $f^*(u_i)$. Hence, every $u_i$ is assigned to a label no later than iteration $i$. □

# 5. ALGORITHM ANALYSIS

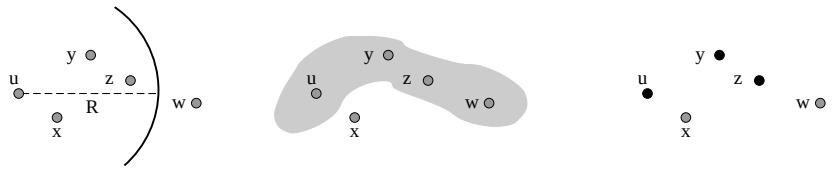
At the heart of the analysis of **ApproxLabel** is a proof that limited dependency between the labels exists, i.e., the label that a vertex receives depends on only a limited number of other labels. This allows us to simultaneously bound both the cost of the solution and the number of vertices assigned to each label. We show that the expected cost of the labeling is at most $O\left(\frac{\log n}{\varepsilon}\right)$ times the value of an optimal solution. We can prove this bound despite the lack of "full" dependency among the labels assigned to vertices (as in [16]). We note that proving this bound requires at least a limited level of dependency. We show that at most $\min\left\{\frac{O(\log k)}{1-\varepsilon}, \ell + 1\right\}(1+\varepsilon)\ell$ vertices are assigned to the same label. We are able to prove this bound, despite the lack of independence between the labels assigned to vertices, via a new inequality of Janson [15] for tail bounds of (partly) dependent random variables.

## 5.1 Approximation Factor

We bound the expected cost of the solution found by **ApproxLabel** in two stages. First, we show that the separation cost of adjacent vertices is at most $O\left(\frac{\ln n}{\varepsilon}\right)$ times the separation cost of these vertices in the relaxation. Second, we show that the expected assignment cost of labels to vertices is at most $1 + \varepsilon$ times the assignment cost of labels to vertices in the relaxation. The general method used for proving these bounds is somewhat similar to a method used in [5], though in our case the analysis is much more involved, since each random subset chosen is constructed from two different random subsets. We use the following notations (as in [5]) to simplify the presentation: $\bar{d}(u, v) = \min\left\{\frac{\varepsilon}{1+\varepsilon}, d(u, v)\right\}$, and $\tau = \frac{1+\varepsilon}{\varepsilon} \cdot \ln n$.

**Expected Separation Cost of Neighboring Vertices.** Define the following events for every $u, v \in V$:

- $A(u, v)$ - Vertices $u$ and $v$ are assigned to labels at different iterations.
- $B_i(u, v)$ - During iteration $i$, exactly one of $u$ and $v$ is assigned to a label.
- $C_i(u, v)$ - At the end of iteration $i$, neither $u$ nor $v$ are assigned to a label.




(i) u,x,y and z are close to u with respect to the metric d.

(ii) u,y,z and w are close to u with respect to u's root label.

(iii) Intersection of both random subsets.

**Figure 2: Iteration with root $u$. Assume $f^*(u)$ is denoted by the black color.**

Consider an iteration $i$ where $u_i$ is the root and in which $\Pr[C_{i-1}(u,v)] > 0$, and assume without loss of generality that $\bar{d}(u_i,u) \leq \bar{d}(u_i,v)$. Therefore,

- $\Pr[B_i(u,v) \mid C_{i-1}(u,v)] = \Pr\left[\bar{d}(u_i,u) \leq R \leq \bar{d}(u_i,v)\right] \cdot \Pr\left[c_{f^*(u_i)}(u_i,u) \geq \alpha\right] + \Pr\left[\bar{d}(u_i,v) \leq R \leq \frac{\varepsilon}{1+\varepsilon}\right] \cdot \Pr\left[\left|\{u,v\} \cap \left\{x \mid c_{f^*(u_i)}(u_i,x) \geq \alpha\right\}\right| = 1\right]$
- $\Pr[C_i(u,v) \mid C_{i-1}(u,v)] = \Pr\left[0 \leq R \leq \bar{d}(u_i,u)\right] + \Pr\left[\bar{d}(u_i,u) \leq R \leq \bar{d}(u_i,v)\right] \cdot \Pr\left[c_{f^*(u_i)}(u_i,u) < \alpha\right] + \Pr\left[\bar{d}(u_i,v) \leq R \leq \frac{\varepsilon}{1+\varepsilon}\right] \cdot \Pr\left[\left|\{u,v\} \cap \left\{x \mid c_{f^*(u_i)}(u_i,x) \geq \alpha\right\}\right| = 0\right].$

LEMMA 6. *For every iteration $1 \leq i \leq n$ and any pair of vertices $u,v \in V$:*

**(a)** $\Pr\left[c_{f^*(u_i)}(u_i,u) \geq \alpha\right] = 1 - d(u_i,u)$.

**(b)** $\Pr\left[\left|\{u,v\} \cap \{x \mid c_{f^*(u_i)}(u_i,x) \geq \alpha\}\right| = 1\right] \leq d(u,v)$.

PROOF. To prove (a), we obtain from Property (7), constraints (3) (closeness constraints), (4) (variation distance), and the fact that $\alpha \sim Unif[0,\varphi(u_i,f^*(u_i))]$:

$$\Pr\left[c_{f^*(u_i)}(u_i,u) \geq \alpha\right] = \sum_{j \in L} \Pr\left[c_{f^*(u_i)}(u_i,u) \geq \alpha \middle| f^*(u_i) = j\right] \cdot \Pr[f^*(u_i) = j] = \sum_{j \in L} \frac{c_j(u_i,u)}{\varphi(u_i,j)} \varphi(u_i,j) = 1 - d(u_i,u).$$

To prove (b), we obtain from Property (7), constraints (3) (closeness constraints), (4) (variation distance), (6) (closeness $\Delta$-inequality), and from $\alpha \sim Unif[0,\varphi(u_i,f^*(u_i))]$:

$$\Pr\left[\left|\{u,v\} \cap \{x \mid c_{f^*(u_i)}(u_i,x) \geq \alpha\}\right| = 1\right] = \sum_{j \in L} \left\{\Pr\left[\left|\{u,v\} \cap \left\{x \mid c_{f^*(u_i)}(u_i,x) \geq \alpha\right\}\right| = 1 \;\middle|\; f^*(u_i) = j\right] \cdot \Pr[f^*(u_i) = j]\right\} = \sum_{j \in L} \frac{|c_j(u_i,u) - c_j(u_i,v)|}{\varphi(u_i,j)} \cdot \varphi(u_i,j) \leq 1 - \sum_{j \in L} c_j(u,v) = d(u,v).$$

□

Clearly, for every two vertices $u,v \in V$, $\Pr[C_0(u,v)] = 1$, $\Pr[C_n(u,v)] = 0$ (since, at the end all vertices are labeled), and if there is an iteration $i^*$ such that $\Pr[C_{i^*}(u,v)] = 0$, then for every iteration $i > i^*$: $\Pr[C_i(u,v)] = 0$. Define for every $u,v \in V$, $i^*(u,v) = \operatorname{argmin}_i \{\Pr[C_i(u,v)] = 0\}$. Obviously, the following reverse recursive formula on the probability $\Pr[A(u,v) \mid C_{i-1}(u,v)]$ holds for every $1 \leq i \leq i^*(u,v) - 1$:

$$\Pr[A(u,v) \mid C_{i-1}(u,v)] = \Pr[B_i(u,v) \mid C_{i-1}(u,v)] + \Pr[C_i(u,v) \mid C_{i-1}(u,v)] \cdot \Pr[A(u,v) \mid C_i(u,v)].$$

The following lemma is crucial for the analysis of the approximation factor as it bounds the expected separation cost of adjacent vertices. We prove this lemma by reverse induction on the number of iterations, starting with iteration $i^*(u,v)$, and using the reverse recursive formula on the probability $\Pr[A(u,v) \mid C_{i-1}(u,v)]$.

LEMMA 7. *For every $u,v \in V$, and every $1 \leq i \leq i^*(u,v)$, $\Pr[A(u,v) \mid C_{i-1}(u,v)] \leq d(u,v)\,(1+\varepsilon+\tau)\left(\frac{2n}{n-1} - \frac{i}{n-1}\right)$.*

PROOF. We prove the lemma by reverse induction on the number of iterations.

*Basis:* Consider iteration $i^*(u,v)$. According to the definition of $i^*(u,v)$, if at the beginning of this iteration neither $u$ nor $v$ are assigned to any label, then at the end of the iteration, at least one of them must be assigned to a label. Assuming without loss of generality that $d(u_{i^*(u,v)},u) \leq d(u_{i^*(u,v)},v)$, it must be the case that $d(u_{i^*(u,v)},u) = 0$. Otherwise, there is a positive probability that $u$ is not assigned to a label in iteration $i^*(u,v)$. Since $d(u_{i^*(u,v)},u) \leq d(u_{i^*(u,v)},v)$, this implies that $\Pr[C_{i^*(u,v)}(u,v)] > 0$, contradicting the definition of $i^*(u,v)$. Therefore,

$$\Pr[A(u,v) \mid C_{i^*(u,v)-1}] = \Pr[0 \leq R \leq \bar{d}(u_{i^*(u,v)},v)] + \Pr[\bar{d}(u_{i^*(u,v)},v) \leq R \leq \frac{\varepsilon}{\varepsilon+1}] \cdot \Pr[c_{f^*(u_{i^*(u,v)})}(u_{i^*(u,v)},v) < \alpha]$$

By Lemma 1, Lemma 6, the fact that $1 - e^{-x} \leq x$, and since $\bar{d}(u_{i^*(u,v)},v) \leq d(u_{i^*(u,v)},v)$, we obtain that:

$$\Pr[A(u,v) \mid C_{i^*(u,v)}(u,v)] \leq d(u_{i^*(u,v)},v) \cdot \frac{n}{n-1} \cdot \left(\tau + e^{-\bar{d}(u_{i^*(u,v)},v)\tau} - \frac{1}{n}\right) \leq d(u_{i^*(u,v)},v) \cdot \frac{n}{n-1} \cdot (1+\varepsilon+\tau).$$

If $d(u_{i^*(u,v)},u) = 0$, then $d(u_{i^*(u,v)},v) = d(u,v)$ from the fact that $d$ satisfies the triangle inequality (see Lemma 2).

Additionally, $i^*(u,v) \leq n$, hence:

$$\mathbb{P}\mathrm{r}[A(u,v) \mid C_{i^*(u,v)-1}(u,v)] \leq d(u,v)\,(1+\varepsilon+\tau)\left(\frac{2n}{n-1} - \frac{i^*(u,v)}{n-1}\right).$$

*Step:* Assume correctness for $i+1$, and assume without loss of generality that $\bar{d}(u_i,u) \leq \bar{d}(u_i,v)$. If $\bar{d}(u_i,u) = \frac{\varepsilon}{1+\varepsilon}$, then $\mathbb{P}\mathrm{r}[B_i(u,v) \mid C_{i-1}(u,v)] = 0$ and $\mathbb{P}\mathrm{r}[C_i(u,v) \mid C_{i-1}(u,v)] = 1$. Therefore, by the recursive formula we obtain that:

$$\mathbb{P}\mathrm{r}[A(u,v) \mid C_{i-1}(u,v)] = \mathbb{P}\mathrm{r}[A(u,v) \mid C_i(u,v)],$$

and the lemma follows by the induction hypothesis for $i+1$. Otherwise, $d(u_i,u) < \frac{\varepsilon}{1+\varepsilon}$. By the induction hypothesis for $i+1$ and the recursive formula:

$$\mathbb{P}\mathrm{r}[A(u,v) \mid C_{i-1}(u,v)] \leq \mathbb{P}\mathrm{r}[B_i(u,v) \mid C_{i-1}(u,v)] + \mathbb{P}\mathrm{r}[C_i(u,v) \mid C_{i-1}(u,v)] d(u,v)\,(1+\varepsilon+\tau)\,\frac{2n-i-1}{n-1}$$

By setting the value of $\mathbb{P}\mathrm{r}[B_i(u,v) \mid C_{i-1}(u,v)]$, setting the value of $\mathbb{P}\mathrm{r}[C_i(u,v) \mid C_{i-1}]$, by Lemma 1, Lemma 6, and by the fact that:

$$\mathbb{P}\mathrm{r}\left[\left|\{u,v\} \cap \left\{x \mid c_{f^*(u_i)}(u_i,x) \geq \alpha\right\}\right| = 0\right] \leq \mathbb{P}\mathrm{r}[c_{f^*(u_i)}(u_i,u) < \alpha],$$

we obtain that:

$$\begin{aligned}
\mathbb{P}\mathrm{r}[A(u,v) \mid C_{i-1}(u,v)] \leq \left(\frac{n}{n-1}\right) \cdot \Bigg\{ & \left(1 - e^{-\left(\bar{d}(u_i,v) - \bar{d}(u_i,u)\right)\tau}\right) e^{-\bar{d}(u_i,u)\tau}\,(1-d(u_i,u)) + \\
& \left(e^{-\bar{d}(u_i,v)\tau} - \frac{1}{n}\right) d(u,v) + \Bigg[ \left(1 - e^{-\bar{d}(u_i,u)\tau}\right) + \\
& \left(1 - e^{-\left(\bar{d}(u_i,v) - \bar{d}(u_i,u)\right)\tau}\right) e^{-\bar{d}(u_i,u)\tau} d(u_i,u) + \\
& \left(e^{-\bar{d}(u_i,v)\tau} - \frac{1}{n}\right) d(u_i,u) \Bigg] \cdot \\
& d(u,v)\,(1+\varepsilon+\tau)\left(\frac{2n}{n-1} - \frac{i+1}{n-1}\right) \Bigg\}.
\end{aligned}$$

Since $1 - e^{-x} \leq x$, and $\bar{d}$ satisfies the triangle inequality (this is true since $d$ satisfies the triangle inequality, Lemma 2), we obtain that:

$$\begin{aligned}
\mathbb{P}\mathrm{r}[A(u,v) \mid C_{i-1}(u,v)] \leq \frac{n}{n-1} \cdot \Bigg\{ & \bar{d}(u,v)\tau e^{-\bar{d}(u_i,u)\tau} \cdot (1-d(u_i,u)) + \\
& \left(e^{-\bar{d}(u_i,v)\tau} - \frac{1}{n}\right) \cdot d(u,v) + \\
& \left[1 - e^{-\bar{d}(u_i,u)\tau} + d(u_i,u) \cdot \left(e^{-\bar{d}(u_i,u)\tau} - \frac{1}{n}\right)\right] \cdot \\
& d(u,v)\,(1+\varepsilon+\tau)\left(\frac{2n}{n-1} - \frac{i+1}{n-1}\right) \Bigg\}.
\end{aligned}$$

Recall that $d(u_i,u) < \frac{\varepsilon}{1+\varepsilon}$, thus: $(1+\varepsilon)\,(1-d(u_i,u)) > 1$. Since $\bar{d}(u,v) \leq d(u,v)$, $\forall u,v \in V$, and $\bar{d}(u_i,u) \leq \bar{d}(u_i,v)$, we can conclude that:

$$\begin{aligned}
\mathbb{P}\mathrm{r}[A(u,v) \mid C_{i-1}(u,v)] \leq & \frac{d(u,v)}{n-1} \cdot \tau \cdot (1-d(u_i,u)) + \\
& \frac{n}{n-1} \cdot d(u,v)\,(1-d(u_i,u)) \cdot \\
& (\tau+1+\varepsilon) \cdot \left(e^{-\bar{d}(u_i,u)\tau} - \frac{1}{n}\right) + \\
& \left[\frac{n}{n-1} d(u_i,u)\left(e^{-\bar{d}(u_i,u)\tau} - \frac{1}{n}\right) + \right. \\
& \left. \frac{n}{n-1}\left(1 - e^{-\bar{d}(u_i,u)\tau}\right)\right] \cdot \\
& d(u,v)\,(1+\varepsilon+\tau)\,\frac{2n-i-1}{n-1}
\end{aligned}$$

Since,

$$\frac{n}{n-1}\left[1 - e^{-\bar{d}(u_i,u)\tau} + d(u_i,u)\left(e^{-\bar{d}(u_i,u)\tau} - \frac{1}{n}\right)\right] \leq 1,$$

and $\tau(1-d(u_i,u)) \leq (1+\varepsilon+\tau)$:

$$\begin{aligned}
\mathbb{P}\mathrm{r}[A(u,v) \mid C_{i-1}(u,v)] \leq & d(u,v)\,(\tau+1+\varepsilon) \cdot \\
& \left[(d(u_i,u)+1-d(u_i,u))\,\frac{n}{n-1}\left(e^{-\bar{d}(u_i,u)\tau} - \frac{1}{n}\right) + \right. \\
& \left. \frac{n}{n-1}\left(1 - e^{-\bar{d}(u_i,u)\tau}\right)\right] + \\
& d(u,v)\left(\frac{n-i}{n-1}\right)(1+\varepsilon+\tau) + \frac{d(u,v)}{n-1}\,(1+\varepsilon+\tau)
\end{aligned}$$

Notice that:

$$\left[(d(u_i,u)+1-d(u_i,u))\,\frac{n}{n-1}\left(e^{-\bar{d}(u_i,u)\tau} - \frac{1}{n}\right) + \frac{n}{n-1}\left(1 - e^{-\bar{d}(u_i,u)\tau}\right)\right] = 1.$$

Therefore,

$$\begin{aligned}
\mathbb{P}\mathrm{r}[A(u,v) \mid C_{i-1}(u,v)] \leq & \\
& d(u,v)\,(\tau+1+\varepsilon)\left[1 + \frac{n-i}{n-1} + \frac{1}{n-1}\right] = \\
& d(u,v)\,(\tau+1+\varepsilon)\left(\frac{2n}{n-1} - \frac{i}{n-1}\right)
\end{aligned}$$

□

**Expected Assignment Cost of Labels to Vertices.** We now bound the assignment cost of labels to vertices. Define the following events for every $v \in V$ and $j \in L$:

- $A(v,j)$ - Vertex $v$ is assigned to label $j$.
- $B_i(v,j)$ - During iteration $i$, $v$ is assigned to label $j$.
- $C_i(v)$ - At the end of iteration $i$, $v$ is not assigned to any label.

Consider an iteration $i$ where $\Pr[C_{i-1}(v)] > 0$. Then,

- $\Pr[B_i(v,j) \mid C_{i-1}(v)] = \Pr\left[\bar{d}(u_i,v) \leq R \leq \frac{\varepsilon}{1+\varepsilon}\right] \cdot \Pr\left[\left(c_{f^*(u_i)}(u_i,v) \geq \alpha\right) \wedge (f^*(u_i) = j)\right]$
- $\Pr[C_i(v) \mid C_{i-1}(v)] = \Pr\left[0 \leq R \leq \bar{d}(u_i,v)\right] + \Pr\left[\bar{d}(u_i,v) \leq R \leq \frac{\varepsilon}{1+\varepsilon}\right] \cdot \Pr\left[c_{f^*(u_i)}(u_i,v) < \alpha\right].$

LEMMA 8. *For every iteration $1 \leq i \leq n$, every label $j \in L$, and every vertex $v \in V$:*

$$\Pr\left[\left(c_{f^*(u_i)}(u_i,v) \geq \alpha\right) \wedge (f^*(u_i) = j)\right] = c_j(u_i,v).$$

PROOF. It follows from Property (7), from constraint (3) (closeness constraints), and the additional fact that the distribution of $\alpha$ is $Unif[0, \varphi(u_i, f^*(u_i))]$, that:

$$\Pr\Big[\left(c_{f^*(u_i)}(u_i,v) \geq \alpha\right) \wedge \left(f^*(u_i) = j\right)\Big] = \Pr\left[c_{f^*(u_i)}(u_i,v) \geq \alpha \mid f^*(u_i) = j\right] \cdot \Pr\left[f^*(u_i) = j\right] = \frac{c_j(u_i,v)}{\varphi(u_i,j)} \cdot \varphi(u_i,j) = c_j(u_i,v).$$

□

Clearly, for every $v \in V$, $\Pr[C_0(v)] = 1$, $\Pr[C_n(v)] = 0$ (since, at the end all vertices are labeled), and if there is an iteration $i^*$ such that $\Pr[C_{i^*}(v)] = 0$, then for every iteration $i > i^*$: $\Pr[C_i(v)] = 0$. Define for every $v \in V$, $i^*(v) = \text{argmin}_i \{\Pr[C_i(v)] = 0\}$. Obviously, the following reverse recursive formula on $\Pr[A(v,j) \mid C_{i-1}(v)]$ holds for every $1 \leq i \leq i^*(v) - 1$:

$$\Pr[A(v,j) \mid C_{i-1}(v)] = \Pr[B_i(v,j) \mid C_{i-1}(v)] + \Pr[C_i(v) \mid C_{i-1}(v)] \cdot \Pr[A(v,j) \mid C_i(v)].$$

The following lemma is crucial for the analysis of the approximation factor as it bounds the expected assignment cost of labels to vertices. We prove this lemma by reverse induction on the number of iterations, starting with iteration $i^*(v)$, and using the reverse recursive formula on the probability $\Pr[A(v,j) \mid C_{i-1}(v)]$.

LEMMA 9. *For every $v \in V$, every $1 \leq i \leq i^*(v)$, and every $j \in L$, $\Pr[A(v,j) \mid C_{i-1}(v)] \leq (1+\varepsilon)\varphi(v,j)$.*

PROOF. We prove the lemma by reverse induction on the number of iterations.
*Basis:* Consider iteration $i^*(v)$. If $v$ is still not assigned to a label at the beginning of this iteration, then $v$ must be assigned to a label during this iteration. Thus, $d(u_{i^*(v)}, v) = 0$ and $c_{j'}(u_{i^*(v)}, v) = \varphi(u_{i^*(v)}, j') = \varphi(v,j')$ for every $j' \in L$. If this is not the case, then there is a positive probability that $v$ is not assigned to a label in iteration $i^*(v)$, implying that $\Pr[C_{i^*}(v)] > 0$, contradicting the definition of $i^*(v)$. Hence, $\Pr[A(v,j) \mid C_{i^*(v)-1}] = \varphi(u_{i^*(v)}, j) = \varphi(v,j) \leq (1+\varepsilon)\varphi(v,j)$.
*Step:* Assume correctness for $i+1$. If $\bar{d}(u_i,v) = \frac{\varepsilon}{1+\varepsilon}$, then $\Pr[B_i(v,j) \mid C_{i-1}(v)] = 0$ and $\Pr[C_i(v) \mid C_{i-1}(v)] = 1$. Therefore, by the recursive formula we obtain that:

$$\Pr[A(v,j) \mid C_{i-1}(v)] = \Pr[A(v,j) \mid C_i(v)],$$

and the lemma follows by the induction hypothesis for $i+1$. Otherwise, $d(u_i,v) < \frac{\varepsilon}{1+\varepsilon}$. By the induction hypothesis for $i+1$ and the recursive formula: $\Pr[A(v,j) \mid C_{i-1}(v)] \leq \Pr[B_i(v,j) \mid C_{i-1}(v)] + \Pr[C_i(v) \mid C_{i-1}(v)] \cdot (1+\varepsilon)\varphi(v,j)$. By setting the value of $\Pr[B_i(v,j) \mid C_{i-1}(v)]$, setting the value of $\Pr[C_i(v) \mid C_{i-1}(v)]$, by Lemma 8, and by Lemma 6, we obtain that:

$$\Pr[A(v,j) \mid C_{i-1}(v)] \leq \Pr\left[\bar{d}(u_i,v) \leq R \leq \frac{\varepsilon}{1+\varepsilon}\right] c_j(u_i,v) + \left\{\Pr\left[\bar{d}(u_i,v) \leq R \leq \frac{\varepsilon}{1+\varepsilon}\right] d(u_i,v) + \Pr[0 \leq R \leq \bar{d}(u_i,v)]\right\}(1+\varepsilon)\varphi(v,j).$$

Recall that $d(u_i,v) < \frac{\varepsilon}{1+\varepsilon}$, thus, $(1+\varepsilon)(1-d(u_i,v)) > 1$. By constraint (3) (closeness constraint), $c_j(u_i,v) \leq \varphi(v,j)$. Therefore, we conclude:

$$\Pr[A(v,j) \mid C_{i-1}(v)] \leq \Pr\left[\bar{d}(u_i,v) \leq R \leq \frac{\varepsilon}{1+\varepsilon}\right](1+\varepsilon)(1-d(u_i,v))\varphi(v,j) + \left\{\Pr\left[\bar{d}(u_i,v) \leq R \leq \frac{\varepsilon}{1+\varepsilon}\right] d(u_i,v) + \Pr[0 \leq R \leq \bar{d}(u_i,v)]\right\} \cdot (1+\varepsilon)\varphi(v,j) = (1+\varepsilon)\varphi(v,j).$$

□

**Completing the Analysis.** Denote by $OPT$ the cost of an optimal labeling of the balanced uniform metric labeling problem.

THEOREM 10. *The expected cost of a solution found by* **ApproxLabel** *is at most:* $\left(\frac{1+\varepsilon}{\varepsilon}\ln n + 1 + \varepsilon\right)\left(\frac{2n}{n-1}\right) \cdot OPT$.

PROOF. Define the following random variables for every $u, v \in V$ and every $j \in L$:

- $Y_{u,v}$ - Indicator for the event that vertices $u$ and $v$ are assigned to different labels by **ApproxLabel**;
- $Y_{v,j}$ - Indicator for the event that $v$ is assigned to label $j$ by **ApproxLabel**.

By Lemma 7,

$$\mathbb{E}[Y_{u,v}] \leq \Pr[A(u,v) | C_0(u,v)] \leq (\tau + 1 + \varepsilon)\left(\frac{2n}{n-1}\right) d(u,v),$$

and by Lemma 9,

$$\mathbb{E}[Y_{v,j}] = \Pr[A(v,j) | C_0(v)] \leq (1+\varepsilon)\,\varphi(v,j).$$

Hence, the expected cost of the solution found by **ApproxLabel** is:

$$\mathbb{E}\left[\sum_{v \in V}\sum_{j \in L} c(v,j)Y_{v,j}\right] + \mathbb{E}\left[\sum_{u,v \in V} w(u,v)Y_{u,v}\right] \leq \left(\frac{1+\varepsilon}{\varepsilon}\ln n + 1 + \varepsilon\right)\left(\frac{2n}{n-1}\right) \cdot \left(\sum_{v \in V}\sum_{j \in L} c(v,j)\varphi(v,j) + \sum_{u,v \in V} w(u,v)d(u,v)\right).$$

The proof is completed using Lemma 3. □

## 5.2 Bounding The Number of Vertices Assigned to a Label

We present two upper bounds on the number of vertices assigned to a single label. Each upper bound corresponds to a different **RootLabel** procedure that satisfies Property (7). Neither of the two upper bounds is strictly better than the other, thus one can take the better of the two bounds for each instance.

As stated earlier, spreading constraints play a crucial role in the proof of an upper bound on the number of vertices that are assigned to the same label. We need the following lemma for both upper bounds, since it gives a lower bound on the diameter of a "large" subset of vertices.

LEMMA 11. *Assume $(V,d)$ is a semi-metric that satisfies the spreading constraints (5). Then, for every $S \subseteq V$ such that $|S| > \gamma\ell$ ($\gamma > 1$) and every $u \in S$, there exists a vertex $v \in S$ such that $d(u,v) > 1 - \frac{1}{\gamma}$.*

PROOF. Let $S \subseteq V$ such that $|S| > \gamma\ell$ ($\gamma > 1$) and $u \in S$. According to constraints (5), $\sum_{v\in S} d(u,v) \geq |S| - \ell$. Therefore, there exists $v \in S$ such that $d(u,v) \geq 1 - \frac{\ell}{|S|}$. Since $|S| > \gamma\ell$, we obtain that $d(u,v) > 1 - \frac{1}{\gamma}$. □

In order to prove the two upper bounds, we need to use the fact that spheres with a bounded diameter do not contain too many vertices. This can be easily derived from the previous lemma.

COROLLARY 12. *For every $v \in V$ and every $\gamma > 1$,*

$$\left|\left\{x | d(v,x) \leq 1 - \frac{1}{\gamma}\right\}\right| \leq \gamma\ell.$$

### 5.2.1 Upper Bound of $\frac{O(\ln k)}{1-\varepsilon}(1+\varepsilon)\ell$

In order to achieve this upper bound, we use the following simple **RootLabel** procedure, that assigns (independently) to each vertex a label according to its distribution over the label set. The random variable $\beta$ is chosen independently in each iteration.

**Algorithm RootLabel**

1. For $i = 1$ to $n$ do:
2. Choose $\beta \sim Unif[0,1]$.
3. Set $f^*(u_i) \leftarrow \min\{j > 0 \mid \sum_{j'=1}^{j} \varphi(u_i, j') \geq \beta\}$.
4. Output $f^*$.

Clearly, Property (7) is maintained in **RootLabel**. Define for every $v \in V$ and every $j \in L$ an indicator $Y_{v,j}$ for the event that at the end of algorithm **ApproxLabel**, vertex $v$ is assigned to label $j$. According to Lemma 9, $\mathbb{E}[Y_{v,j}] \leq (1+\varepsilon)\varphi(v,j)$. By constraints (2) (capacity constraints), we obtain that $\mathbb{E}\left[\sum_{v\in V} Y_{v,j}\right] \leq (1+\varepsilon)\ell$.

If $\{Y_{v,j}\}_{v\in V}$ were independent random variables for every $j$, a simple Chernoff bound would suffice to achieve a good bound on the number of vertices assigned to each label. However, $\{Y_{v,j}\}_{v\in V}$ are clearly dependent. Thus, we use a more delicate bound by showing that $\{Y_{v,j}\}_{v\in V}$ are only "partly" dependent. This is done by showing that the maximum degree in the dependency graph of $\{Y_{v,j}\}_{v\in V}$ is not too "large".

LEMMA 13. *For every $v \in V$ and every $j \in L$, $Y_{v,j}$ is independent of $\left\{Y_{u,j} \mid d(u,v) > 2 \cdot \frac{\varepsilon}{1+\varepsilon}\right\}$.*

PROOF. Recall that $\{f^*(u_i)\}_{1\leq i\leq n}$ are independent and all $R$'s and $\alpha$'s chosen by **ApproxLabel** are independent. Therefore, the (random) intersections of the two random subsets found by **ApproxLabel** in each iteration are also independent. Thus, if $u, v \in V$ cannot be assigned to a label in the same iteration, $f(u)$ and $f(v)$ are independent. Since $\left\{u | d(u,v) > 2\frac{\varepsilon}{1+\varepsilon}\right\}$ is contained in the set of all vertices that cannot be assigned to a label in the same iteration as $v$ (regardless of the order of the roots), the proof is completed. □

We use the following definition of Janson [15] for a *dependency graph* for the random variables $\{Y_{v,j}\}_{v\in V}$, for every $j \in L$. In fact, this definition is equivalent to the one used in the Lovász Local Lemma [1].

DEFINITION 2. *A graph $\Gamma = (V', E')$ is a dependency graph for $\{Y_{v,j}\}_{v\in V}$, if $V' = \{Y_{v,j}\}_{v\in V}$ and for every $V'' \subset V'$ and $Y_{v,j} \notin V''$ such that $Y_{v,j}$ is not connected by any edge in $E'$ to vertices in $V''$, $Y_{v,j}$ is independent of $\{Y_{v,j}\}_{v\in V''}$.*

Let us define a graph as follows:

$$\Gamma_j = \left(\{Y_{v,j}\}_{v\in V}, \left\{(Y_{u,j}, Y_{v,j}) | d(u,v) \leq 2\frac{\varepsilon}{1+\varepsilon}\right\}\right).$$

Clearly, by the above definition of a dependency graph and by Lemma 13, $\Gamma_j$ is a dependency graph for $\{Y_{v,j}\}_{v\in V}$. Recall that $\varepsilon < 1$, hence by Corollary 12, the maximum degree of $\Gamma_j$ is at most $\frac{(1+\varepsilon)\ell}{1-\varepsilon} - 1$. Let us denote the maximum degree of $\Gamma_j$ by $\Delta_j$. Janson [15] proved the following inequality for every $t > 0$:

$$\mathbb{P}\mathrm{r}\left\{\sum_{v\in V} Y_{v,j} \geq \mathbb{E}\left[\sum_{v\in V} Y_{v,j}\right] + t\right\} \leq \exp\left(-\frac{8t^2}{25(\Delta_j+1)\left(\sum_{v\in V}\mathrm{Var}(Y_{v,j}) + \frac{bt}{3}\right)}\right),$$

where $b$ satisfies the following condition: $Y_{v,j} - \mathbb{E}[Y_{v,j}] \leq b$, for every $v \in V$.

It is important to notice that the above inequality is useful for random variables having a small variance, which is the case for us. Since for every $v \in V$, $Y_{v,j}$ is an indicator and $\mathbb{E}\left[\sum_{v\in V} Y_{v,j}\right] \leq (1+\varepsilon)\ell$, we get that $b = 1$ and $\sum_{v\in V}\mathrm{Var}(Y_{v,j}) \leq (1+\varepsilon)\ell$. Hence, we get that if we choose $t = \delta(1+\varepsilon)\ell$, the following holds:

$$\mathbb{P}\mathrm{r}\left[\sum_{v\in V} Y_{v,j} \geq (1+\delta)(1+\varepsilon)\ell\right] \leq \exp\left(-\frac{8(1-\varepsilon)\delta^2}{25\left(1+\frac{\delta}{3}\right)}\right).$$

We need the above probability to be at most $\frac{1}{2k}$. Therefore, it suffices to choose $\delta = \frac{O(\ln k)}{1-\varepsilon}$. Using the union bound over all $k$ labels, we can conclude that with probability at least $\frac{1}{2}$, there are at most $(1+\delta)(1+\varepsilon)\ell = \frac{O(\ln k)}{1-\varepsilon}(1+\varepsilon)\ell$ vertices assigned to each label.

### 5.2.2 Upper Bound of $(\ell+1)(1+\varepsilon)\ell$

In order to achieve this upper bound, we interpret the probabilities $\{\varphi(u)\}_{u\in V}$ as an $n \times k$ matrix where the row that corresponds to $v$ is $\varphi(v)$. We denote this matrix by $\Phi$, and additionally we denote by an upper index $i$, the $i$-th column of a matrix. The next procedure generates a new $n \times n$ matrix, denoted by $A$, which is doubly stochastic (the

sum of each row and column is equal to 1). This is done by going over the columns of $\Phi$ according to their index, and "aggregating" and "splitting" them, such that the sum of each new column is precisely 1. This process can be done, since the sum of all the elements in $\Phi$ is $n$ (the sum of each row is 1 and there are $n$ rows).

After constructing $A$, we sample a random permutation matrix $A^*$ from a Birkhoff decomposition of $A$. A Birkhoff decomposition of a doubly-stochastic matrix $A$ is a convex finite combination of permutation matrices that is equal to $A$. Specifically,

$$A = \sum_{i=1}^{N} \lambda_i A_i,$$

where $\sum_{i=1}^{N} \lambda_i = 1$, and for every $1 \leq i \leq N$, $\lambda_i \geq 0$, and $A_i$ is a permutation matrix. It is well known that such a decomposition always exists and it can be constructed in polynomial time. One can view $\{\lambda_i\}_{1 \leq i \leq N}$ as a distribution over the permutation matrices that appear in the decomposition. Hence, when we "sample" from a Birkhoff decomposition, we choose a permutation matrix according to this distribution.

We use the permutation matrix $A^*$ to sample a label for each vertex in the following way. Since $A^*$ is a permutation matrix, for every $u_i \in V$ there is a unique column $j'$ in $A^*$ such that $A^*(u_i, j') = 1$. We choose a random label from all labels that were "aggregated" into column $j'$ according to their relative value in column $j'$ with respect to vertex $u_i$.

We note that Property (7) holds. Additionally, it is easy to see that there are at most $\ell + 1$ vertices assigned to a single label in $f^*$. This is true since each column $\Phi^i$ has a total value of at most $\ell$ (this is due to constraints (2), the capacity constraints). Thus, each column of $\Phi$ can be "spread" on at most $\ell + 1$ columns of $A$. Therefore, there will always be at most $\ell + 1$ vertices assigned to a single label in $f^*$.

In a single iteration of **ApproxLabel**, all vertices that are assigned to a label are at distance of at most $\frac{\varepsilon}{1+\varepsilon}$ from that iteration's root. According to Corollary 12, there are at most $(1+\varepsilon)\ell$ such vertices. Therefore, we obtain an upper bound of $(\ell+1)(1+\varepsilon)\ell$ on the number of vertices that are assigned to the same label.

The following theorem states the upper bound on the number of vertices that **ApproxLabel** assigns to each label. It follows directly by choosing the best upper bound from the two presented.

THEOREM 14. **ApproxLabel** *finds a solution in which there are at most* $\min\left\{\frac{O(\ln k)}{1-\varepsilon}, \ell+1\right\}(1+\varepsilon)\ell$ *vertices assigned to each label, with probability of at least* $\frac{1}{2}$.

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