Lecture 4

Duality

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4.1 Weak duality

Given a linear program, write all the equalities with a greater than or equal sign. The constraints can be strict, and you can have two types of variables to distinguish. The last equation is not really a constraint.

$$
\begin{aligned}
\text{Min } \sum_{i=1}^{n} c_i x_i & \\
\forall j = 1 \ldots k_1 \qquad & \sum_i a_{ij} x_i \geq b_j \\
\forall j = k_1 + 1 \ldots m \qquad & \sum_i a_{ij} x_i = b_j \\
\forall i = 1 \ldots k_2 \qquad & x_i \geq 0 \\
\forall i = k_2 + 1 \ldots n \qquad & x_i <> 0 | 0
\end{aligned}
$$

For the dual, first create dual variables for each constraint, as though multiplying by some coefficient. Don't create dual variables for the positive constraints. Min becomes max, constraints become variables, variables become constraints, nonnegatives become inequalities, and unconstrained variables become equalities:

$$
\begin{aligned}
\text{Max } \sum_{j=1}^{m} b_j y_j & \\
\forall j = 1 \ldots k_1 \qquad & y_j \geq 0 \\
\forall j = k_1 + 1 \ldots m \qquad & y_j <> 0 \\
\forall i = 1 \ldots k_2 \qquad & \sum_j a_{ij} y_j \leq c_i \\
\forall i = k_2 + 1 \ldots n \qquad & \sum_j a_{ij} y_j = c_i
\end{aligned}
$$

Theorem 4.1. ***Weak Duality*** *Take any feasible primal solution X and take any feasible dual solution Y.*

Then the objective function of the primal is greater than or equal to the objective function of the dual:

$$\sum_{i=1}^{n} c_i x_i \geq \sum_{j=1}^{m} b_j y_j$$

Proof. Proof uses constraints of primal and dual.

$$\begin{aligned}
\sum_{i=1}^{n} c_i x_i &= \sum_{i=1}^{k_2} c_i x_i + \sum_{i=k_2+1}^{n} c_i x_i \\
&= \sum_{i=1}^{k_2} c_i x_i + \sum_{i=k_2+1}^{n} \left(\sum_{j=1}^{m} a_{ij} y_j \right) x_i \\
& \quad \text{[the } x_i\text{'s are nonnegative, so if the } c_i\text{'s decrease, the sum will decrease]} \\
&\geq \sum_{i=1}^{k_2} \left(\sum_{j=1}^{m} a_{ij} y_j \right) x_i + \sum_{i=k_2+1}^{n} \left(\sum_{j=1}^{m} a_{ij} y_j \right) x_i \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} y_j x_i \\
&= \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij} x_i y_j \\
&\geq \sum_{j=1}^{m} b_j y_j
\end{aligned}$$

□

4.2 Strong Duality

Theorem 4.2. ***Strong Duality*** *Optimum objective functions are equal.*

Proof. We will prove: if the objective function is optimum, then the dual is greater than or equal to the primal. With weak duality, this gives us equality.

Write the primal constraints as

$$\forall j = 1 \ldots k_1 \qquad \sum_i a_{ij} x_i \geq b_j \tag{4.1}$$

$$\forall j = k_1 + 1 \ldots m \qquad \sum_i a_{ij} x_i \geq b_j \tag{4.2}$$

$$\forall j = k_1 + 1 \ldots m \qquad \sum_i (-a_{ij}) x_i \geq -b_j \tag{4.3}$$

$$\forall i = 1 \ldots k_2 \qquad x_i \geq 0 \tag{4.4}$$

Equations (2) and (3) are equivalent to the original strict equality constraint. Call the variable associated with (1) $y_j \geq 0$; with (2), $z_j \geq 0$; with (3), $w_j \geq 0$; and with (4), $v_i \geq 0$. Assume that min $\sum_{i=1}^{n} c_i x_i = d$ is optimum. Then the primal constraints must imply $\sum_{i=1}^{n} c_i x_i \geq d$.

By Farka's lemma, there must be a proof of this fact. On the left hand side, we should get something by summing that is $\geq d$. This will prove our theorem.

$$c_i = \sum_{j=1}^{k_1} a_{ij} y_j + \sum_{j=k_1+1}^{m} a_{ij} z_j + \sum_{j=k_1+1}^{m} (-a_{ij}) w_j + v_i (i = 1 \ldots k_2)$$

Right hand side:

$$\sum_{j=1}^{k_1} b_j y_j + \sum_{j=k_1+1}^{m} b_j z_j + \sum_{j=k_1+1}^{m} (-b_j) w_j \geq d$$

We can ignore the v_i's by making it $\geq$, and tighten up the equation:

$$\sum_{j=1}^{k_1} b_j y_j + \sum_{j=k_1+1}^{m} b_j (z_j - w_j) \geq d$$

So there exists a solution that is feasible for the dual and is greater than or equal to the primal. However, this proof assumes the existence of feasible solutions; without them, the solution is unbounded. The equations can't both be infeasible. If one is infeasible, Farka's gives you a feasible solution in the dual. $\square$

4.3 Complementary Slackness

Definition 4.1. Primal Complementary Slackness

$$\forall i = 1 \ldots k_2 c_i x_i = \sum_{j=1}^{m} a_{ij} y_j x_i$$

Either $x_i = 0$ or $x_i > 0$ and $c_i = \sum_{j=1}^{m} aij y_j$.

Definition 4.2. Dual Complementary Slackness

$$\forall i = j \ldots k_1 b_j y_j = \sum_{i=1}^{n} a_{ij} x_i y_j$$

Either $y_j = 0$ or $y_j > 0$ and $b_j = \sum_{i=1}^{n} aij x_i$.

If the objective functions are equal then the complementary slackness conditions hold; if both complementary slackness conditions hold, then the objective functions are equal.

4.3.1 Integer Program

$$\begin{aligned} \text{Min } \sum_{i=1}^{n} c_i x_i & \\ \forall j = 1 \ldots m \qquad & \sum_{i=1}^{n} a_{ij} x_i \geq b_j \\ x_i \in \{0,1\} & \\ \text{All } c_i, b_j, a_{ij} \geq 0 & \end{aligned}$$

Relaxation: $x_i \geq 0$
Dual:

$$\begin{aligned} \text{Max } \sum_{j=1}^{m} b_j y_j & \\ \forall i = 1 \ldots n \qquad & \sum_{j=1}^{m} a_{ij} y_j \leq c_i \\ y_j \geq 0 & \end{aligned}$$

We are not interested in the typical optimum solution. Instead, we try to get an integral solution for the primal and some nice solution for the dual that we can prove is close to optimum.

Any arbitrary integral solution $\geq$ OPT$_I \geq$ OPT$_{LP}$ =OPT$_{Dual} \geq$ Value$_{Feasibledual}$

Take any feasible integral solution. It is greater than or equal to any feasible dual. The feasible integral solution that we find with the LP is less than or equal to some factor times the feasible dual that the algorithm finds.

Either $x_i = 0$ or $x_i > 0$ and $\sum_{j=1}^{n} a_{ij} y_j \geq c_i$. This is sufficient for complementary slackness. This may not happen, so ideally at least

$$\alpha \left(\sum_{j=1}^{n} a_{ij} y_j \right) \geq c_i \text{ for some } \alpha \geq 1$$

This is called α-approximate primal complementary slackness. Similarly, β-approximate dual complementary slackness conditions are: either $y_j = 0$ or $y_j > 0$ and $\sum_{i=1}^{n} a_{ij} x_i \leq \beta b_j$. Primal feasibility already implies $\sum a_{ij} x_i \geq b_j$, so $\beta \geq 1$.

α-β factor algorithm: primal complementary slackness satisfied by α, dual complementary slackness satisfied by β.

Intuitively, x_i's are resources. y_j's are money you pay. If some resource is bought then you almost pay correctly in dual solution. Constraints are people and they are independent. Everyone wants to satisfy his or her constraint. y_j is the amount of money that the jth person is willing to spend, but he doesn't want to overpay. If the price of the resource is c_i, no one wants to overpay for that resource. Complementary slackness is saying no resource is underpaid, either.

4.3.2 Vertex cover example

Given an undirected graph G, each node v has weight function $w_v \geq 0$. The objective is to pick vertices $v_1, \ldots v_k$ such that the total weight is as small as possible and all edges are covered by at least one vertex: $\forall e = v_i, v_j$, either v_i is picked or v_j is picked, or both. We claim there is an algorithm with factor 2.

LP: Let x_v be 1 if it's picked and 0 otherwise.

$$
\begin{aligned}
\text{Min } \sum_v w_v x_v & \\
\forall e = u, v \qquad & x_u + x_v \geq 1 \\
x_v \in \{0, 1\} \qquad & \text{relax to } x_v \geq 0
\end{aligned}
$$

Dual:

$$
\begin{aligned}
\text{Max } \sum_{e \in \delta(u)} y_e & \\
& \delta(u) = \{e = \{u, v\}\} \\
\forall u \qquad & \sum_{e \in \delta(u)} y_e \leq w_u \\
\forall v \qquad & \sum_{e \in \delta(u)} y_e \leq w_v
\end{aligned}
$$

Take any maximal dual: if you try to increase any y_e, you'll get an infeasible solution. Take any maximal y_e and pick corresponding vertex. This is feasible, because if you look at any edge, it must have been picked on at least one side. y_e is in two inequalities, and once has no slack (since solution is maximal), so one is picked. The factor 2 comes because $\beta = 2$. There is no slack in the primal, the b_js are 1. But you may have picked two vertices.

Intuitively: every edge is offering y_e money to get chosen. A vertex that gets enough money will sell; it may sell to two y_e's and that's the factor of 2.