Lecture 3

Polar Duality and Farkas' Lemma

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3.1 Polytope $\implies$ bounded polyhedron

Last lecture, we were attempting to prove the Minkowsky-Weyl Theorem: every polytope is a bounded polyhedron, and every bounded polyhedron is a polytope. The second direction (every bounded polyhedron is a polytope) was shown last lecture, using an argument about corner points. This lecture, we show that every polytope is a bounded polyhedron by investigating the concept of *polar duality*.

Theorem 3.1. $\textsc{ConvexHull}(w_1, \ldots, w_k) = P =$ *Polytope* $\implies$ *P is a bounded polyhedron*

Assume that P is full-dimensional, and, WLOG, 0 is in the interior of P. (This latter requirement can simply be seen as a normalization, since it is easily accomplished by translation.)

From this, we can deduce that there must be some ball that fits inside P:

$$\exists r > 0 \text{ s.t. } B(0, r) \subset P$$

We now define the *polar dual* of a polytope P, denoted $P*$:

Definition 3.1. The *polar dual* of a set P, denoted P^*, is the set $\{y | y^T x \leq 1, \forall x \in P\}$.

When P is a polytope, as in this case, the following defintion is equivalent:

$$P^* = \{y | y^T w_i \leq 1, i = 1 \ldots k\}$$

It is easy to see that these two definitions are equivalent, because $\forall x \in P$, $x = \lambda_1 w_1 + \ldots + \lambda_k w_k, \lambda_i \geq 0$. Therefore, $y^T x = \sum_{i=1}^k \lambda_i y^T \cdot w_i \leq \sum_{i=1}^k \lambda_i = 1$.

Lemma 3.2. *P^* is a bounded polyhedron.*

Proof. $r \frac{y}{||y||} \in B(0, r) \subset P$. So $\frac{ry}{||y||} \in P$. Thus, $y^T \left(\frac{ry}{||y||} \right) \leq 1$. Simplifying, $\frac{||y||^2}{||y||} r \leq 1$, so $||y|| \leq \frac{1}{r}$. Since the length of y is bounded, P^* is a bounded polyhedron. $\square$

We still need to show that $(P^*)^*$ is a bounded polyhedron and $P^{**} = P$. Once we've proven that, we've proven that any polytope P is a bounded polyhedron, so we're done.

The following argument is tempting, but wrong. Note that:

$$\forall x \in P, \forall y \in P^* x \cdot y \leq 1$$

Flipping this around, we see:

$$\forall y \in P^*, \forall x \in P x \cdot y \leq 1$$

This looks a lot like the requirement for membership in P^{**}! Unfortunately, this intuition is wrong if P isn't convex. It is possible to find $S = \{w_1, w_2, \ldots, w_k\}$ such that $S^* = P^*$, so $S^{**} = P^{**} = P \neq S$! (When S is non-convex.)

Here's an alternate approach that does work: we show that $P \subset P^{**}$ and $P^{**} \subset P$.

Proof. The first direction is easy: consider $x \in P$. For any $y \in P^*$, $x \cdot y \leq 1$. Therefore, $x \in P^{**}$ as well, since the only requirement is that $y \cdot x \leq 1$, which was already ensured.

For the second direction, we wish to show that for $x \notin P$, $x \notin P^{**}$. Let C, δ define a hyperplane separating the polytope from x: $C^T \cdot z < \delta \forall z \in P$, and $C^T \cdot x > \delta$. Since $0 \in P$, $C^T \cdot 0 < \delta$, implying that $\delta > 0$. WLOG, let $\delta = 1$.

We have: $C^T \cdot z < 1 \forall z \in P$. So by the definition of P^*, $C \in P^*$. Since $C^T x > 1$ and $C \in P^*$, $x \notin P^{**}$. $\square$

3.2 Homework

3.2.1 Homework #1

Construct polynomial time algorithms for the following:

1. A simple polygon is one that has no self-intersections (two edges that cross) or self-touching (an edge that passes through a vertex, or two vertices with the same coordinate). Given an ordered list of points, $p_1, p_2, \ldots, p_n$, determine if the polygon they define, $\text{POLYGON}(p_1, p_2, \ldots, p_n, p_1)$ is simple.

2. Given a simple polygon $\text{POLYGON}(p_1, p_2, \ldots, p_n, p_1)$, determine if a point z is in the polygon or not.

3. Find a triangulation of a simple polygon, $\text{POLYGON}(p_1, p_2, \ldots, p_n, p_1)$. A triangulation is a set of triangles, $T_1, T_2, \ldots, T_k$ such that:

   (a) $T_1 \cup T_2 \cup \ldots \cup T_k = \text{POLYGON}(p_1, p_2, \ldots, p_n, p_1)$

   (b) $interior(T_i) \cap interior(T_j) = \emptyset, \forall i \neq j$

   (c) $\forall i vertices(T_i) \subset p_1, p_2, \ldots, p_n$

3.2.2 Homework #2

Define a function POLARDUALITY: $P \rightarrow P^*$. Determine whether this function is a bijection for the following domains:

1. Set of all closed convex sets containing 0.
2. Set of all closed convex sets containing 0 in the interior.
3. Set of all polyhedra containing 0.
4. Set of all polyhedra containing 0 in the interior.
5. Set of all polytopes containing 0.
6. Set of all polytopes containing 0 in the interior.

3.3 Alternate Proof of Farkas' Lemma

3.3.1 Homogeneous case

Theorem 3.3. $a_1^T x \geq 0, a_2^T x \geq 0, \ldots, a_m^T \geq 0 \implies C^T x \geq 0$ *iff* $C = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m \forall i : \lambda_i \geq 0$.

We first present a few useful definitions.

Definition 3.2. A *cone* is a convex set such that, $\forall x \in \mathrm{C}one$, $\lambda x \in \mathrm{C}one \forall \lambda \geq 0$. Alternately, a *cone* is an intersection of half-spaces defined by hyperplanes passing through the origin.

Definition 3.3. A *polyhedral cone* is a cone defined by a finite number of hyperplanes.

A cone may be finitely generated by points $x_1, x_2, \ldots, x_k$ as follows:

$$\textsc{Cone}(x_1, x_2, \ldots x_k) = y | y = \sum_{i=1}^{k} \lambda_i x_i, \forall i : \lambda_i \geq 0$$

Proof of Farkas' Lemma. The first direction is easy:

If $C = \lambda_i a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m \forall i : \lambda_i \geq 0$, then $C^T x = \sum_i \lambda_i a_i^T x \geq 0$.

For the other direction, we show that if $C \neq \lambda_1 a_1 + \cdots + \lambda_m a_m$, then $C^T x < 0$. Equivalently, if $C \notin \textsc{Cone}(a_1, a_2, \ldots, a_m)$ then $C^T x < 0$.

Let $d\delta$ be a vector such that:

1. $d^T z > \delta \forall z \in \textsc{Cone}(a_1, a_2, \ldots, a_m)$
2. $d^T C < \delta$

Since $0 \in \textsc{Cone}(a_1, a_2, \ldots, a_m)$, $d^T 0 > \delta$. WLOG, let $\delta = -1$.

Now we have that $d^T z > -1$ and $d^T C < -1$.

We know, therefore, that $d^T a_1 > -1$. Therefore, $\frac{1}{\epsilon} a_1 \in \textsc{Cone}(a_1, a_2, \ldots, a_m)$, so $d^T(\frac{1}{\epsilon} a_1) > -1$. Multiplying both sides by ϵ, we have: $d^T a_1 > \epsilon$.

In the limit, $d^T a_1 \geq 0$, since this inequality is true for *all* epsilon. Therefore, we have a d^T such that $d^T a_1 \geq 0, d^T a_2 \geq 0, \ldots, d^T a_m \geq 0$ but $C^T d < 0$. □

3.3.2 Non-homogeneous case

Theorem 3.4. $a_1^T x \geq b_1, a_2^T x \geq b_2, \ldots, a_m^T \geq b_m \implies C^T x \geq d$ iff $C = \sum_{i=1}^m \lambda_i a_i, \lambda_i \geq 0$ and $d \leq \sum_{i=1}^m \lambda_i b_i$

Remark. As the notetaker, I could not follow this argument. My notes reflect this, and my write-up reflects my notes. Therefore, I recommend looking at Schrijver's notes on combinatorial optimization, which contain an alternate proof of this theorem.

Proof. As a helpful step, we show that for any $z \geq 0$, $a_1^T x - b_1 z \geq 0, \ldots a_m^T x - b_m z \geq 0 \implies C^T x \geq dz$. If we can show this, then we simply apply the homogeneous version and we're done.

Case 1: $z > 0$

Case 2: $z = 0$, so $a_1^T x \geq 0 \ldots a_m^T x \geq 0$
$x_1 + \lambda x$, $\lambda \geq 0$

$C^T(x_1 + \lambda x) \geq d$, so $C^T x \geq 0$.

Consider the $m+1$ dimension vector $(C, -d) = \lambda_1(a_1, -b_1) + \lambda_2(a_2, -b_2) + \cdots + \lambda_m(a_m, -b_m) + \lambda_{m+1}(0, 1)$

$C = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m$
$-d = -b_1\lambda_1 - b_2\lambda_2 + \cdots + -b_m\lambda_m + \lambda_{m+1}$
$-d \geq -(b_1\lambda_1 + b_2\lambda_2 + \cdots + b_m\lambda_m)$
$d \leq b_1\lambda_1 + b_2\lambda_2 + \cdots + b_m\lambda_m$

□

3.4 Applications of Farkas' Lemma

Remark. As a notetaker, I didn't understand all of this, either.

Consider a business owned by $N = \{1, 2, \ldots, n\}$ partners. The profit they make is $P(N)$. Any subset of them $S \subset N$ working together could make a profit of $P(S)$. Therefore, in dividing up the profits, each subset S must receive at least $P(S)$ or they would have incentive to go off and start their own business.

In other words, a solution to this profit-dividing problem (if one exists) must meet the following criteria: $P(N) = P_1 + P_2 + \cdots + P_n$ such that $\forall S, \sum_{i \in S} P_i \geq P(S)$. P_i in this case is the amount of profit that goes to the ith partner.

Definition 3.4. The *core* of this game is a set of all solutions such that $P(S) \geq 0$ and $P(T) \geq P(S)$ for any $T \supset S$.

Definition 3.5. In a *balanced game*, there exists a fractional decomposition $N = \lambda_1 S_1 + \lambda_2 S_2 + \cdots + \lambda_k S_k$, where $\forall j \sum_{i:j \in S_i} \lambda_i = 1$ and $P(N) \geq \sum_{i=1}^{k} \lambda_i P(S_i)$.

Theorem 3.5 (Bondareva-Shapley). *The core is non-empty if and only if the game is* balanced.

Proof. First, we show that a core implies a balanced game. $P(N) = P_1 + P_2 + \cdots + P_n$.

$$\sum_{i=1}^{k} \lambda_i P(S_i) \leq \sum_{i=1}^{k} \lambda_i \sum_{j \in S_i} P_j$$

. We can reorder the sums to obtain:

$$\sum_j P_j \sum_{i:j \in S_i} \lambda_i = \sum_j P_j = P(N)$$

In the reverse direction, we show that an empty core implies an imbalanced game.

$$-(P_1 + P_2 + \cdots + P_n) \geq -P(N) \rightarrow \lambda$$

$$\forall S \sum_{i \in S} P_i \geq P(S) \rightarrow \lambda_S$$

$$\forall i - \lambda + \sum_{S:i \in S} \lambda_S = 0\ ;\ \lambda, \lambda_S \geq 0$$

$$-\lambda P(N) + \sum \lambda_i P(S) > 0\ ;\ \lambda > 0$$

Then just apply Farkas' lemma. □