CSE 522
Linear Algebraic Applications in Theory

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Relationships between a graph and its spectrum

Let A be an adjacency matrix for a graph G. Here, we investigate the relationship between the spectrum of A (eigenvalues) and the properties of the graph G.

Assume G is an undirected graph so that A is symmetric. The Laplacian matrix $L(G)$ is defined as $D - A$ where

$$D = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & d_n \end{bmatrix}$$

and d_i is the degree of node i.

Alternatively, the Laplacian can be expressed as follows:

$$L(u,v) = \begin{cases} d_i & \text{if } u = v \\ -1 & \text{if } u \text{ and } v \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

Let $P = D^{-1}A$ so that

$$p_{ij} = \begin{cases} \frac{1}{d_i} & \text{if } (i,j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

then P corresponds to a random walk Markov model on the graph G with adjacency matrix A.

Relationships with graph properties

- Use of eigenvectors for graph partitioning
- Algorithmic results for random graphs (e.g. coloring, independent set)
- Relationships with the vertex expansion and the edge expansion of a graph

  The vertex expansion of a graph G is expressed as follows

  $$c_V(G) = \min_{|X| \leq \frac{n}{2}} \frac{|N(X) - X|}{|X|}$$

  where $X \subset V$ and $|N(X) - X|$ is the number of neighboring vetices of the set X not counting any vertices in X.
  Similarly, the edge expansion of the graph G is given by

  $$c_E(G) = \min_{|X| \leq \frac{n}{2}} \frac{E(X, \overline{X})}{|X|}$$

  where $E(X, \overline{X})$ is the number of edges across the cut between X and $\overline{X}$.

  C_V and C_E are intimately related to the **mixing rate** of a Markov model - "how long till we're in a random place in the graph?", or rather, "how long till the Markov Chain reaches a stationary distribution?" The *min* limits how quickly you get to a "random" place in the graph. The edge expansion is also important (as we will later see) as it relates to the **conductance** of the Markov Chain. Further, it can be shown that the vertex and edge expansions can be bounded as follows:

  $$\Omega(\frac{\lambda}{d}) \leq c_V(G) \leq O(\sqrt{\lambda})$$

  $$\frac{\lambda}{2} \leq c_E(G) \leq \sqrt{2d\lambda}$$

  where d is the maximum degree of the graph and λ is the second smallest eigenvalue of the Laplacian of the graph.

Perron-Frobenius Theorem for nonnegative matrices: *If A is a strictly positive matrix then*

- *one of its eigenvalues, λ_0, is positive and greater than (in absolute value) all other eigenvalues*
- *there is a positive eigenvector corresponding to that eigenvalue λ_0*
- *the dimension of the eigenmanifold corresponding to λ_0 is 1 (i.e., $dim(B_{\lambda_0}) = 1$)*

The same result holds when all entries of A are nonnegative (instead of strictly positive) and $\exists m > 0$ such that A^m is a positive matrix. If A is the adjancency matrix of a graph G, these conditions imply G is connected and has an odd cycle (not bipartite).

Facts that follow from the theorem

If G is a regular graph of degree d, then d is the largest eigenvalue of A and

$$A\vec{e} = d\vec{e}$$

where $\vec{e} = (1, 1, 1, ..., 1)$. Further, the only eigenvector which has all positive components is $\vec{e}$ since all other eigenvectors are orthogonal to it.

Recall that $P = D^{-1}A$ so that

$$\sum_j p_{ij} = 1$$

This means that if A is not regular then P is not symmetric.

$$P\vec{e} = \vec{e}$$

$$\vec{\pi}P = \vec{\pi}$$

Here, the distribution $\vec{\pi}$ is the stationary distribution of the Markov Chain. $\vec{\pi}$ is an eigenvector of the matrix P with corresponding eigenvalue 1.

$$\pi_i = \sum_j \pi_j p_{ij} \qquad \forall i$$

If A is the adjacency matrix for a regular graph with degree d then we can express the relationship between the eigenvalues and eigenvectors of P and $D - A$ as follows

$$\begin{aligned} Px = \lambda x &\Rightarrow D^{-1}Ax = \lambda x \\ &\Rightarrow Ax = \lambda Dx \end{aligned}$$

$$(D - A)x = dx - \lambda dx = d(1 - \lambda)x$$

So that if $\vec{x}$ is an eigenvector of P then $\vec{x}$ is an eigenvector of $D - A$ with eigenvalue $d(1 - \lambda)$.

If we order the eigenvalues of P, $\lambda_n \leq \lambda_{n-1} \leq \ldots \leq \lambda_2 < \lambda_1 = 1$, then the largest eigenvalue of P, $\lambda_1 = 1$, corresponds to the smallest eigenvalue of $D - A$ which equals zero, and $0 < d(1 - \lambda_2)$ where λ_2 is the second smallest eigenvalue of $D - A$.

Theorem(Chung)**:** *Let G be a $(d - 1)$ regular, connected graph with adjacency matrix A. Let $A' = A + I$, and let $\lambda_{max} = max(|\lambda_2|, |\lambda_n|)$. Then $\lambda_1 = d$ and $|\lambda_{max}| < d$ and the diameter of G is*

$$diam(G) \leq \left\lceil \frac{log(n-1)}{log(\frac{d}{\lambda_{max}})} \right\rceil$$

Proof: We have

$$diam(G) = min\{m | (A')^m \text{has all nonzero entries}\}$$

Let $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n}$ be the orthonormal set of eigenvectors for A'. Then the largest eigenvalue of A', namely λ_1, has value d and its corresponding eigenvector is $\vec{e_1} = \frac{\vec{e}}{\sqrt{n}}$. We can then decompose A' so that

$$A' = \Psi \Lambda \Psi^{-1}$$

which means

$$\begin{aligned} (A')^m &= \Psi \Lambda^m \Psi^{-1} \\ &= \sum \lambda_i^m \vec{u_i} \vec{u_i}^T \end{aligned}$$

This gives us

$$
\begin{aligned}
(A')^m_{(r,s)} &= \sum_{1\le i\le n} \lambda_i^m u_{ir} u_{is} \\
&\ge \frac{d^m}{n} - \left| \sum_{i>1} \lambda_i^m u_{ir} u_{is} \right| \\
&\ge \frac{d^m}{n} - |\lambda_{max}|^m \sum_{i>1} |u_{ir}||u_{is}| \\
&\ge \frac{d^m}{n} - |\lambda_{max}|^m \left(\sum_{i>1} u_{ir}^2 \right)^{\frac{1}{2}} \left(\sum_{i>1} u_{is}^2 \right)^{\frac{1}{2}} \qquad \text{by Cauchy-Schwartz} \\
&\ge \frac{d^m}{n} - |\lambda_{max}|^m \left(1 - \frac{1}{n} \right) \qquad \text{since all } \vec{u} \text{ are orthonormal} \\
&> 0 \qquad \text{if} \quad \left(\frac{d}{\lambda_{max}} \right)^m > n-1
\end{aligned}
$$

Thus, $\left(\frac{d}{\lambda_{max}}\right)^m = n-1$ yields the theorem's result. □