University of Washington
Department of Computer Science and Engineering
CSE 521, Spring 1998

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Homework 7, **Due May 20, 1998**

Problem 1. Dr. Spock:

Page 627, Problem 27-3.

Problem 2. Maximum Flow by Scaling:

Page 627, Problem 27-5.

Problem 3. Bidirectional flow:

A *bidirected* graph has two types of edges, directed edges $u \to v$ and source edges $u \leftrightarrow v$. The (unit-capacity) bidirectional flow problem is given a bidirected graph and a distinguished vertex t, find a maximum flow to t. The directed edges $u \to v$ can carry one unit of flow from u to v. A source edge $u \leftrightarrow v$ introduces the same amount of flow at both u and v, so in the unit capacity case, either one unit of flow is introduced at both u and v or no flow is introduced at u and v.

The matching problem is: given an undirected graph $G = (V, E)$, find a maximum cardinality set of edges $M \subseteq E$. (The graph is not necessarily bipartite.)

a) Show that the matching probem can be solved with an algorithm for unit capacity bidirectional flow.

b) (Optional) Show that the unit capacity bidirectional flow problem can be solved with an algorithm for maximum cardinality matching.

Problem 4. Disjoint paths problem:

Let $G = (V, E)$ be an undirected graph and let $U \subseteq V$. Show that the problem of finding a maximum cardinality set of vertex disjoint paths with endpoints in U can be expressed as a bidirectional flow problem. We wish to find a set P of paths, containing as many paths as possible, such that each path has its endpoints in the set U and distinct paths of P do not share any vertices.

Problem 5. Algebraic techniques for matching:

The bipartite matching problem can be expressed in the following, seemingly bizarre manner: Let $G = (U, V, E)$ be a bipartite graph with $|U| = |V| = n$. Construct an $n \times n$ matrix M which has:

$$m_{ij} = \begin{cases} x_{ij} & \text{if } (u_i, v_j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the matrix entries are either variables of the form x_{ij} or 0.

Show that the determinant of M is zero if and only if G has a perfect matching. The determinant of M is a polynomial in $O(n^2)$ variables.

For this problem, all you need to remember about determinants is the formula for evaluating them:

$$\det(M) = \sum_{\sigma} (-1)^{\operatorname{sgn} \sigma} \prod_{i} m_{i\sigma_i}.$$

Equivalently, you can use the minor expansion of the determinant (see page 736 of the text).

The reason that this result is interesting, is that it leads to an efficient randomized algorithm for testing if a graph has a matching. Evaluating a determinant with the symbolic entries is hard since the expressions can get too large. A trick is to plug random values in for the variables. With high probability, if the polynomial is non-zero, the result of evaluating it randomly will also be non-zero.