CSE 521: Design and Analysis of Algorithms **Fall 2022**

Problem Set 5

Deadline: Dec 9th in gradescope

1) We say a d-regular graph G is an ϵ-expander if $\sigma_2(A/d) \leq \epsilon$, where A is the adjacency matrix of G.

   a) Given a d-regular ϵ-expander G with n vertices prove that for every set $S \subseteq V$ of size $|S| \leq \epsilon^2 n$, $|N(S)| \geq \frac{|S|}{c \cdot \epsilon^2}$ for some universal constant c (independent of n or $|S|$).
   Hint: In this part you can use the following expander mixing lemma:
   Theorem: Let G be a d-regular. Then, for any two sets S, T we have

   $$|E(S,T) - d \cdot \frac{|S| \cdot |T|}{n}| \leq \sigma_2(A)\sqrt{|S| \cdot |T|}$$

   where $|E(S,T)|$ is the number of edges between S and T (counting edges contained in the intersection of S and T twice). In other words, the above theorem implies that the number of edges between S, T is almost the same as the number of such edges in a random d-regular graph.

   b) It can be shown that a random d-regular graph is with high probability a $\frac{2}{\sqrt{d}}$-expander. Use the previous part to show that in such a graph there are $\Omega(n)$ vertices at shortest path distance at most $O(\log_d n)$ of any vertex $v \in V$.

2) Consider the following job scheduling problem: We have a set of m machines $M_1, \dots, M_m$ and a set of n jobs $J_1, \dots, J_n$. For each job J_j we are given a set S_j of machines that can execute this job. It takes t_j seconds to run this job (on any machine) and this job must be run continuously (with no preemption) on a single machine. The load of a machine is the sum of length of all jobs executed on that machine. The goal is assign these jobs to all machines to minimize the maximum load among all machines.

   a) Let $0 \leq x_{i,j} \leq t_j$ be the fraction of job J_j that is executed on the machine M_i. Write a linear programming relaxation for this problem using these variables. Show that if all variables $x_{i,j}$ are restricted to be in $\{0, t_j\}$ then the optimum solution of your program is equal to the optimum solution of the problem.

   b) **Extra Credit:** Let x^* be the optimum vertex solution of this linear program. Construct a bipartite graph $G(M, J)$ with one side corresponding to all jobs and the other side corresponding to all machines where a job J_j is connected to a machine M_i if $x^*_{i,j} > 0$. Prove that this graph is acyclic.

   c) Use the result of the previous part to run the following rounding algorithm: Since $G(M, J)$ is acyclic it is a forest; make each connected component of G rooted at some machine. Assign every leaf job to its unique parent. Assign every non-leaf job to one of its children arbitrarily. Prove that this gives a 2-approximation algorithm.