Summer School on Hashing
Theory and Application

Basics of hashing: k-independence and applications

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Agenda

- Load balancing using hashing
  - Analysis using bounded independence
- Implementation of small independence
- Case studies:
  - Approximate membership
  - Hashing with linear probing
- **Exercise**: Space-efficient linear probing

Prerequisites

- I assume you are familiar with the notions of:
  - a hash table
  - modular arithmetic [and perhaps finite fields]
  - expected value of a random variable

You can read about these things in e.g. CLRS or http://www.daimi.au.dk/~bromille/Notes/un.pdf

Load balancing by hashing

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  Distribute an unknown, possibly dynamic, set *S* of items approximately evenly to a set of buckets.

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- **Goal**:
  Distribute an unknown, possibly dynamic, set *S* of items approximately evenly to a set of buckets.
- **Examples**: Hash tables, SSDs, distributed key-value stores, distributed computation, network routing, parallel algorithms, …
- **Main tool**: Random choice of assignment.

n items into n buckets

- **Assume for now:** Items are placed uniformly and independently in buckets.
- What is the probability that k items end up in one particular bucket?
- Use union bound to get an upper bound:

$$\binom{n}{k} n^{-k} < (n^k/k!)n^{-k} < 1/k!$$

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Conclusion: Probability of having some bucket with k items is at most $n/k!$

$\Rightarrow$ largest bucket has size O(log n/log log n) whp.

n items into *r* buckets

- Use better bound on binomial coefficients:

$$\binom{n}{k} < (en/k)^k$$

- Upper bound, *k* items in particular bucket:

$$\sum_{K \subseteq S,\, |K|=k} r^{-k} < (en/k)^k r^{-k} = (en/kr)^k$$

n items into r buckets

- Use better bound on binomial coefficients:

$$\binom{n}{k} < (en/k)^k$$

Conclusion: If $k > 2en/r > 2\log r$ the probability of k items in any single bucket is $< 1/r$.

- Upper bound, k items in particular bucket:

$$\sum_{K \subseteq S,\, |K|=k} r^{-k} < (en/k)^k r^{-k} = (en/kr)^k$$

k-independence

- **Observation**: Proofs only used probabilities of events involving *k* items.
- **Consequence**: It suffices that the hash function used “behaves fully randomly” when considering sets of *k* hash values.

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Definition: A random hash function h is k-independent if for all choices of distinct $x_1,\ldots,x_k$ the values $h(x_1),\ldots,h(x_k)$ are independent.

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- How do you implement k-independent hashing?

Polynomial hashing

- Random polynomial degree *k*-1 hash function (assuming key *x* from field *F):*

$$p(x) = \sum_{i=0}^{k-1} a_i x^i$$

Polynomial hashing

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k-independent!
Why?

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Map to smaller range in any "balanced" way

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Map to smaller range in any "balanced" way

- Divide-and-conquer Horner's rule:

$$p(x) = x p_{\mathrm{odd}}(x^2) + p_{\mathrm{even}}(x^2)$$

Reduces data dependencies!

Implementing field operations

work by Tobias Christiani

- For GF(2^{64}): Use new CLMUL instruction with sparse irreducible polynomial.
  - Time for k-independence ca. $3k$ ns
- For GF(p), p=2^{61}-1 (Mersenne prime): Use double 64-bit registers and special code for modulo.
  - Time for k-independence ca. k ns

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Fastest known for keys of 61 bits up to more than 100-independence

Tomorrow: Double tabulation

Mikkel Thorup on Danish TV

2-independence

- Degree 1 polynomial: $h(\mathrm{x}) = (ax+b \bmod p) \bmod r$

- Property: 2-independent

  $\Rightarrow$ If $a \neq 0$: collision probability $\leq 1/r$

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- For set S of n elements, $\mathrm{x} \notin S$: $\Pr[h(x) \in h(S)] \leq n/r$.

- Probability of no collisions is $\geq 1 - n^2/r$.

Can map to (say) 128-bit "signature" with extremely small risk of collision

Storing a *set* of signatures

- From last slide:
  For set S of n elements, x $\notin S$: $\Pr[h(x) \in h(S)] \leq n/r$.

- Suppose r=2n and we store $h(S)$ as a bitmap.

0110010110111000110111101010101001100010010111010

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Allows us to determine if $x \in S$ with "false positive" error probability $1/2$.

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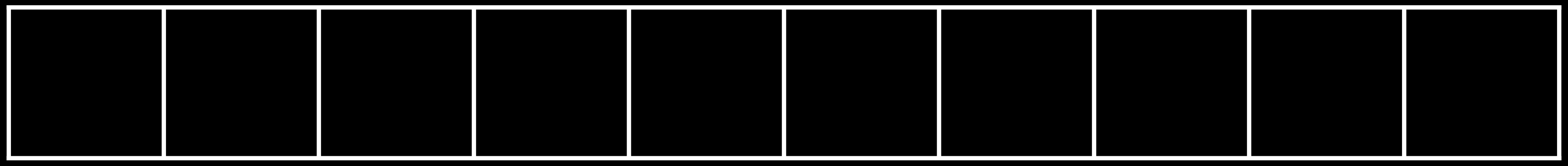
Space 2 bits/item

Allows us to determine if $x \in S$ with "false positive" error probability 1/2.

Linear probing

- A simple method for placing a set of items into a hash table.
- No pointers, just keys and vacant space.
- One of the first hash tables invented, still practically important.

Hashing with linear probing



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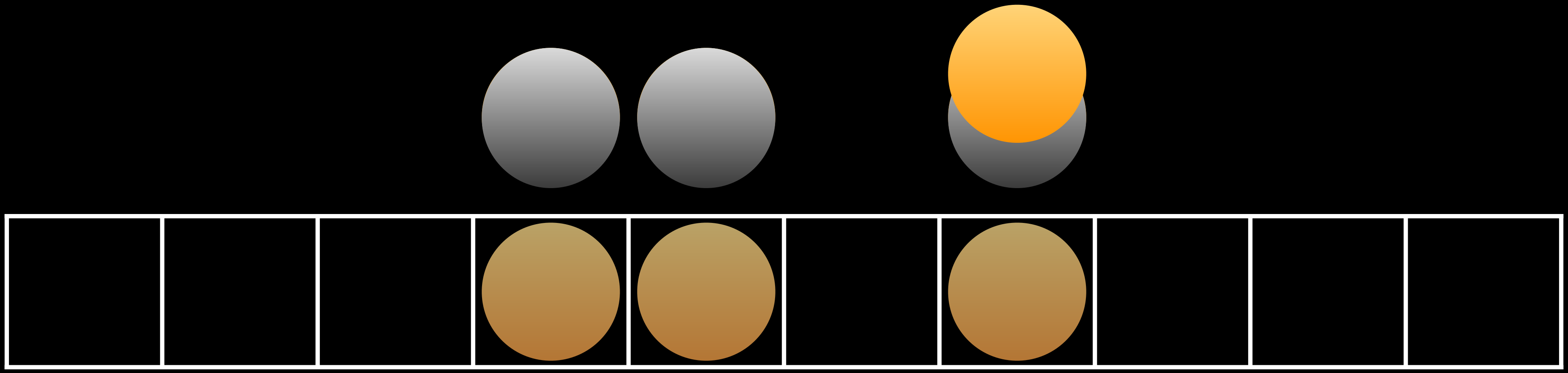

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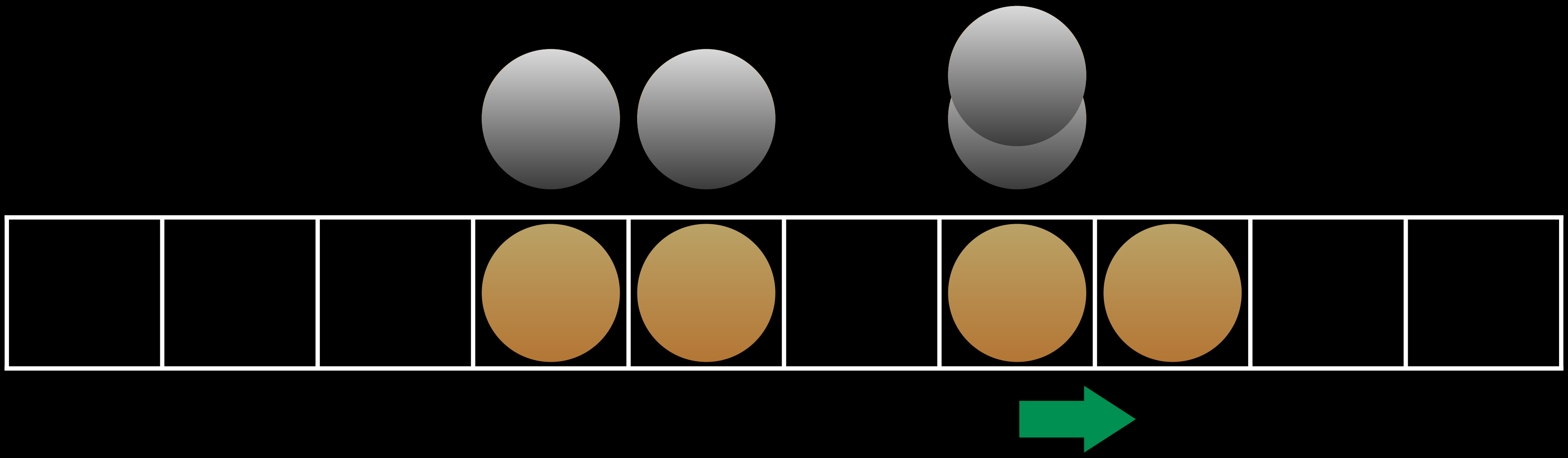

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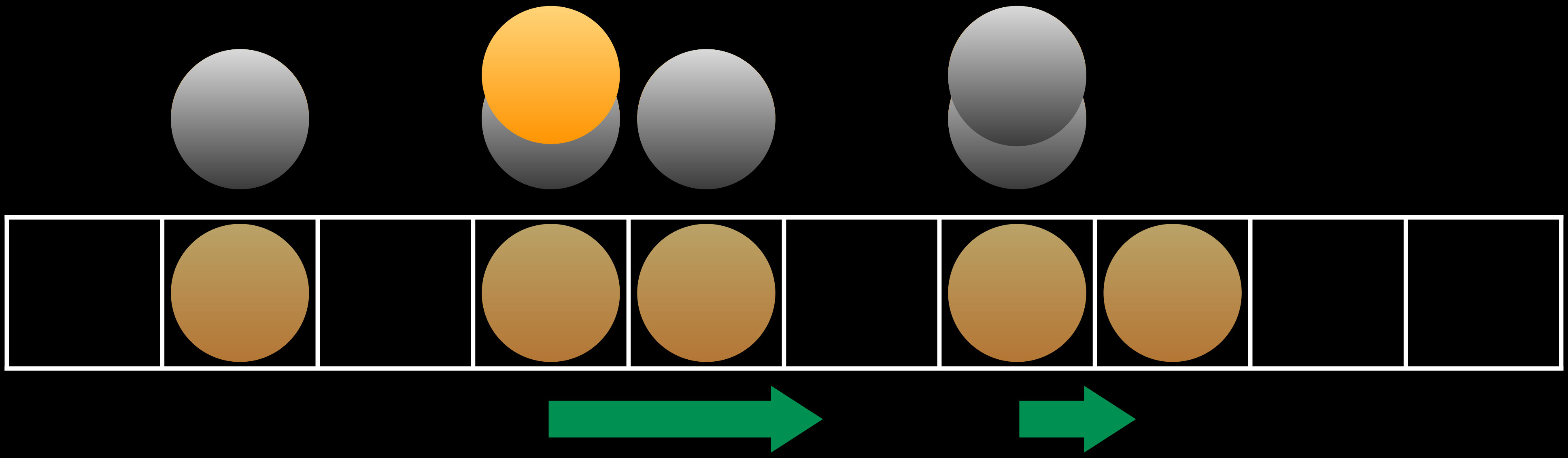

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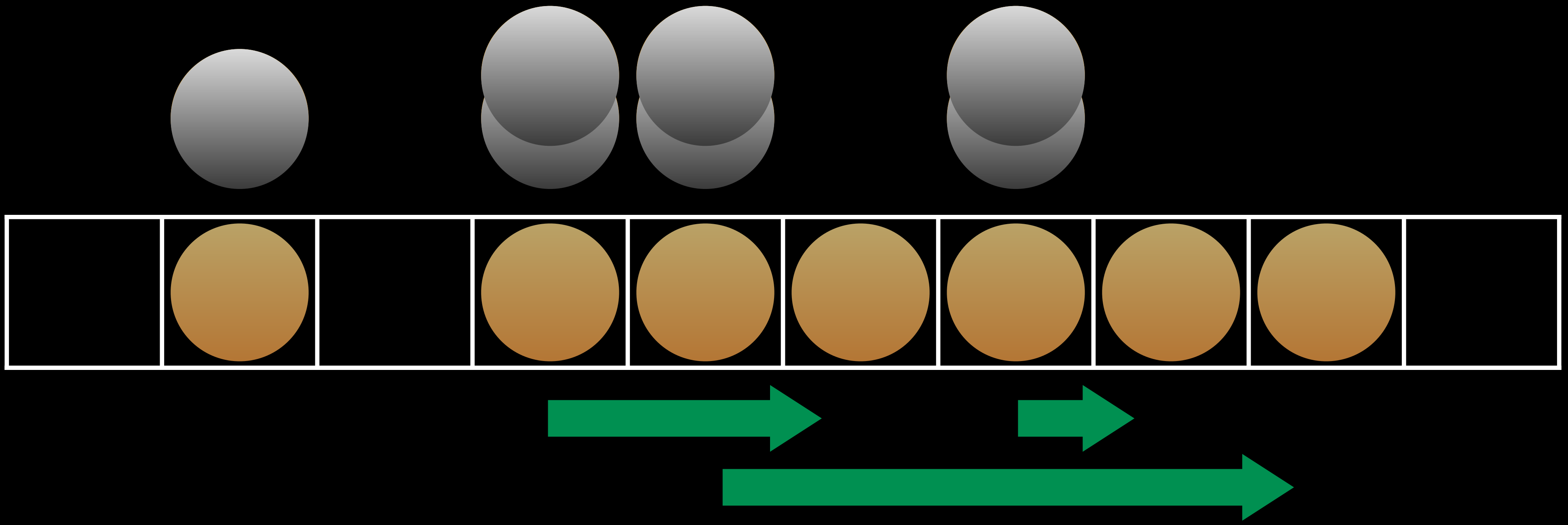
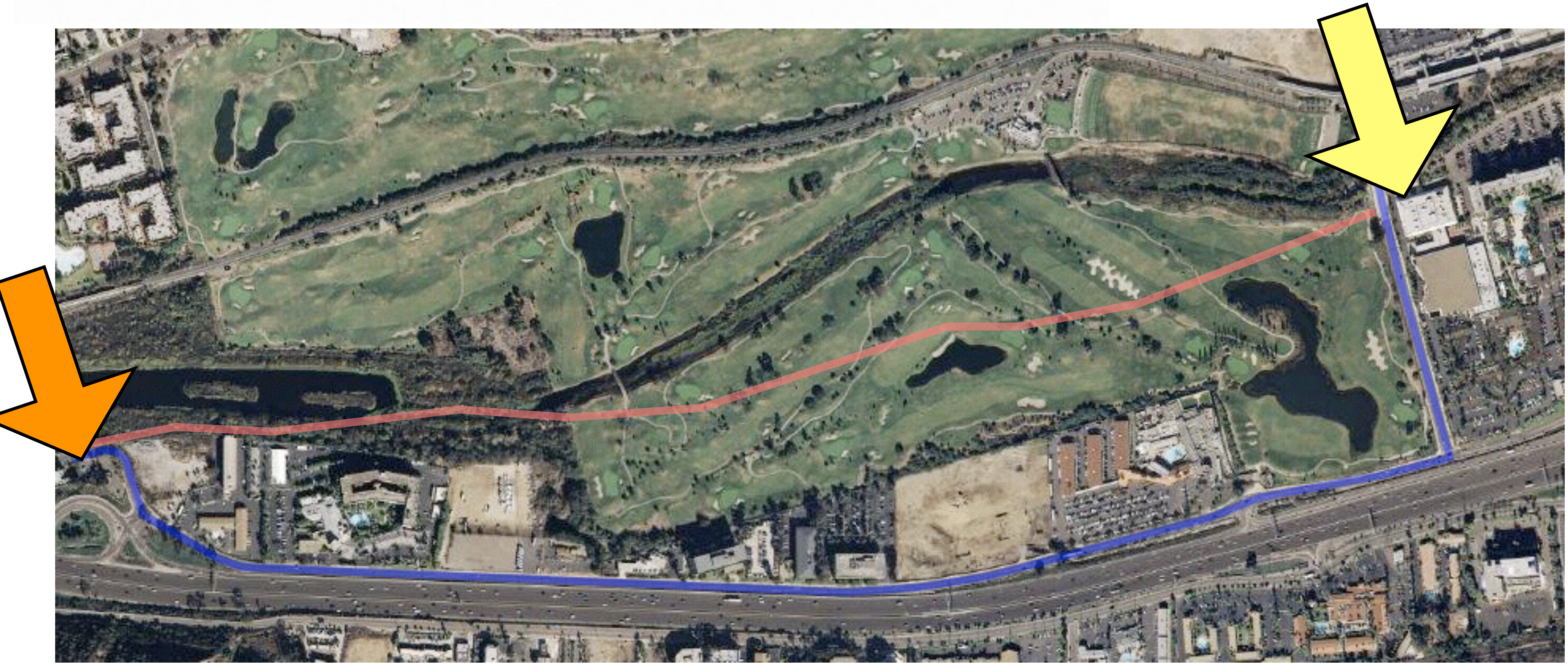


389 km/h

20 km/h

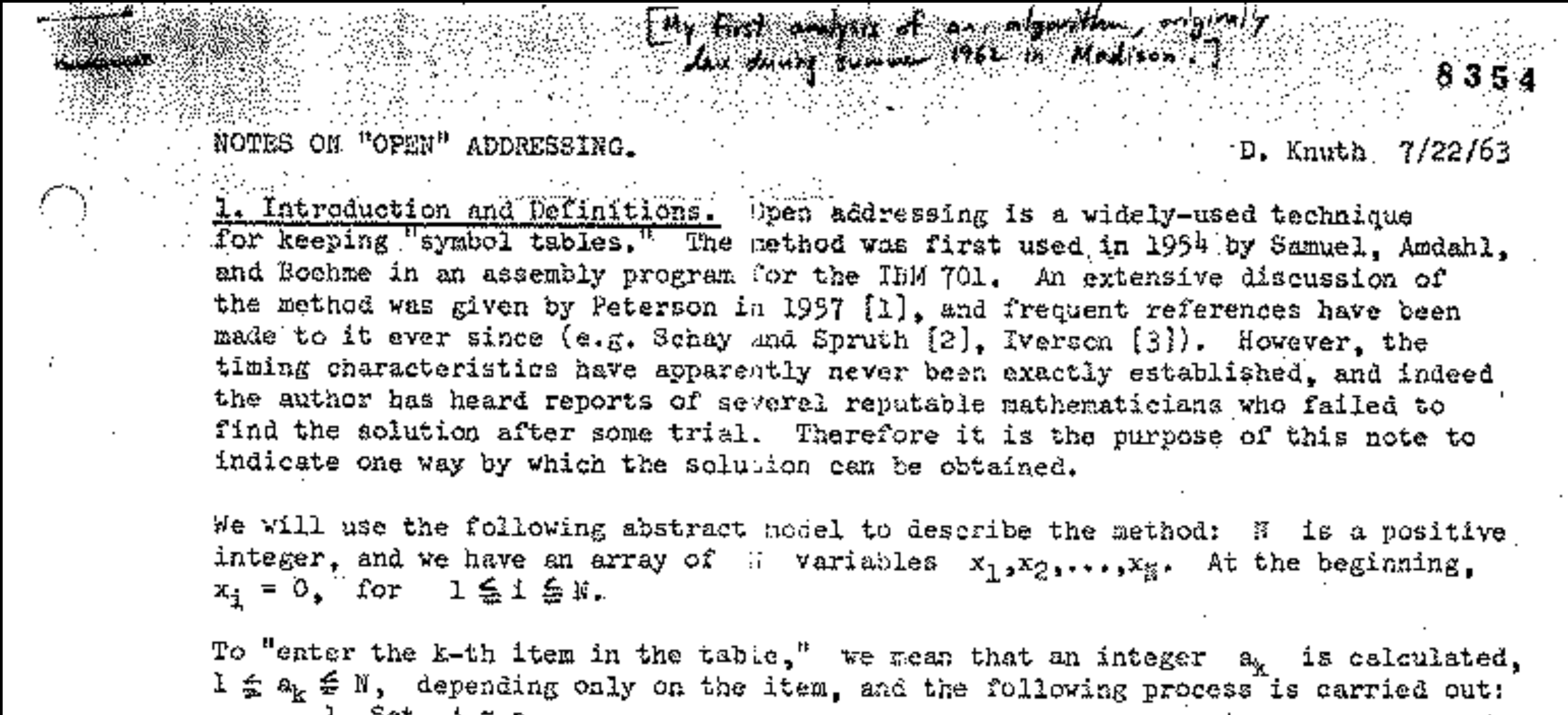


Race car vs golf car

- Linear probing uses a sequential scan and is thus *cache-friendly*.
- Order of magnitude speed difference between sequential and random access!

History of linear probing

- First described in 1954.
- Analyzed in 1962 by D. Knuth, aged 24. Assumes hash function h is fully random.



[My first analysis of an algorithm, originally done during summer 1962 in Madison.]

8354

NOTES ON "OPEN" ADDRESSING. D. Knuth 7/22/63

1. Introduction and Definitions. Open addressing is a widely-used technique for keeping "symbol tables." The method was first used in 1954 by Samuel, Amdahl, and Boehme in an assembly program for the IBM 701. An extensive discussion of the method was given by Peterson in 1957 [1], and frequent references have been made to it ever since (e.g. Schay and Spruth [2], Iverson [3]). However, the timing characteristics have apparently never been exactly established, and indeed the author has heard reports of several reputable mathematicians who failed to find the solution after some trial. Therefore it is the purpose of this note to indicate one way by which the solution can be obtained.

We will use the following abstract model to describe the method: N is a positive integer, and we have an array of N variables $x_1, x_2, \ldots, x_N$. At the beginning, $x_i = 0$, for $1 \leq i \leq N$.

To "enter the k-th item in the table," we mean that an integer a_k is calculated, $1 \leq a_k \leq N$, depending only on the item, and the following process is carried out:

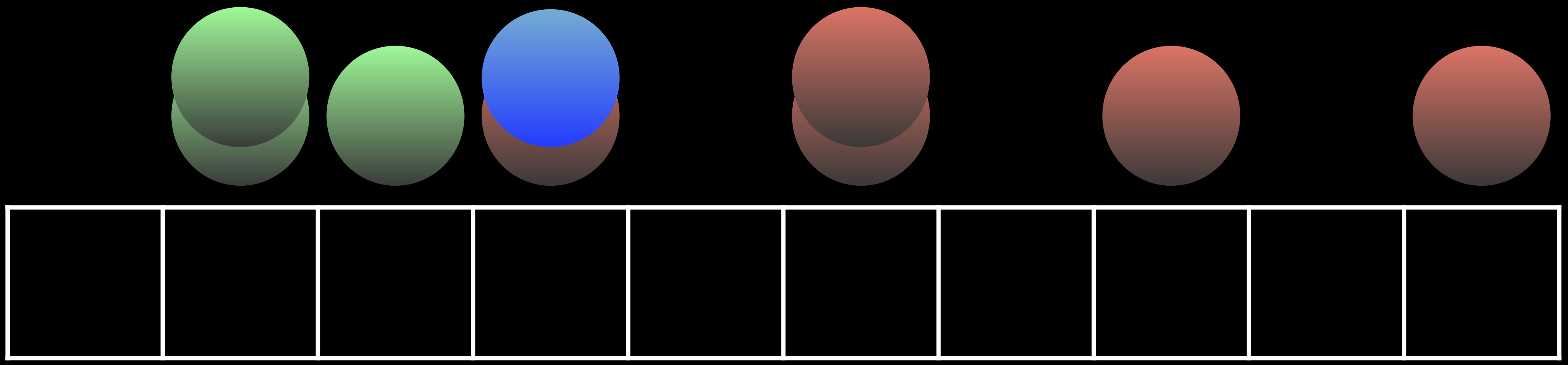
1. Set

History of linear probing

- First described in 1954.
- Analyzed in 1962 by D. Knuth, aged 24. Assumes hash function h is fully random.
- Over 30 papers using this assumption.
- **Since 2007**: We know simple, efficient hash functions that make linear probing provably work!

Modern proof

- Idea: Link the number of steps used to insert an item x to the size of intervals around $h(x)$ being "full" of hash values.



<u>Notation:</u> $L_I = |\{x \in S \mid h(x) \in I\}|$

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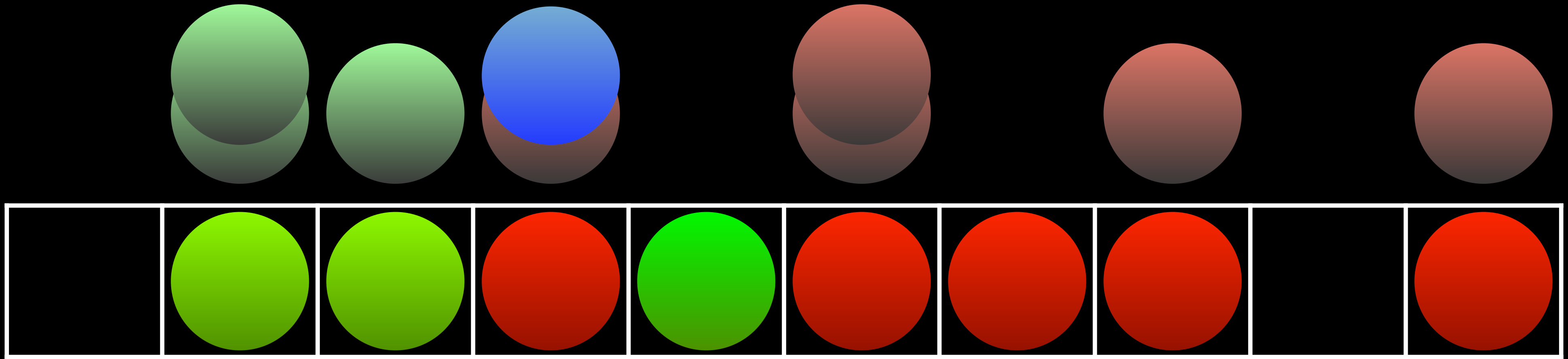

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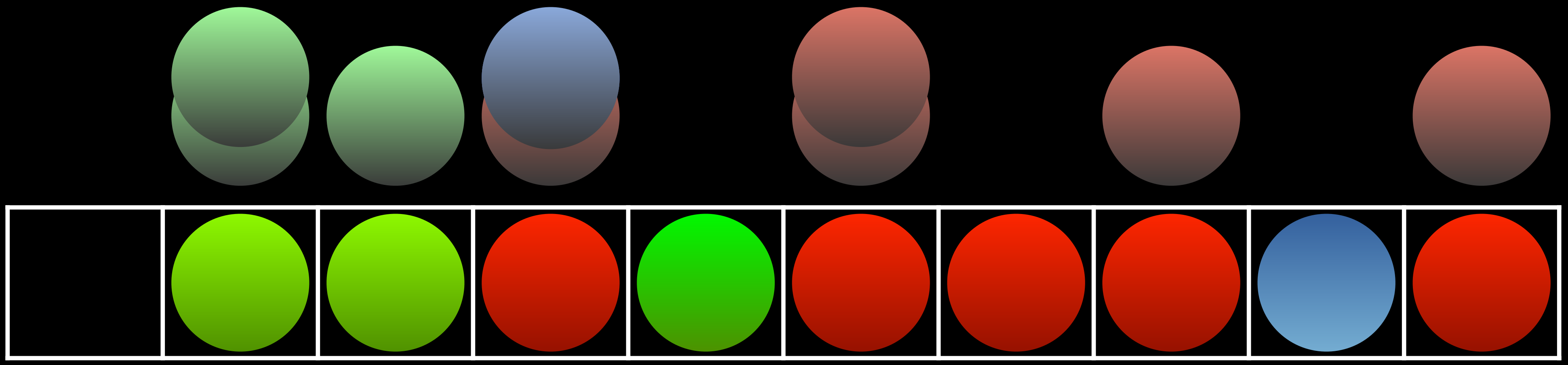
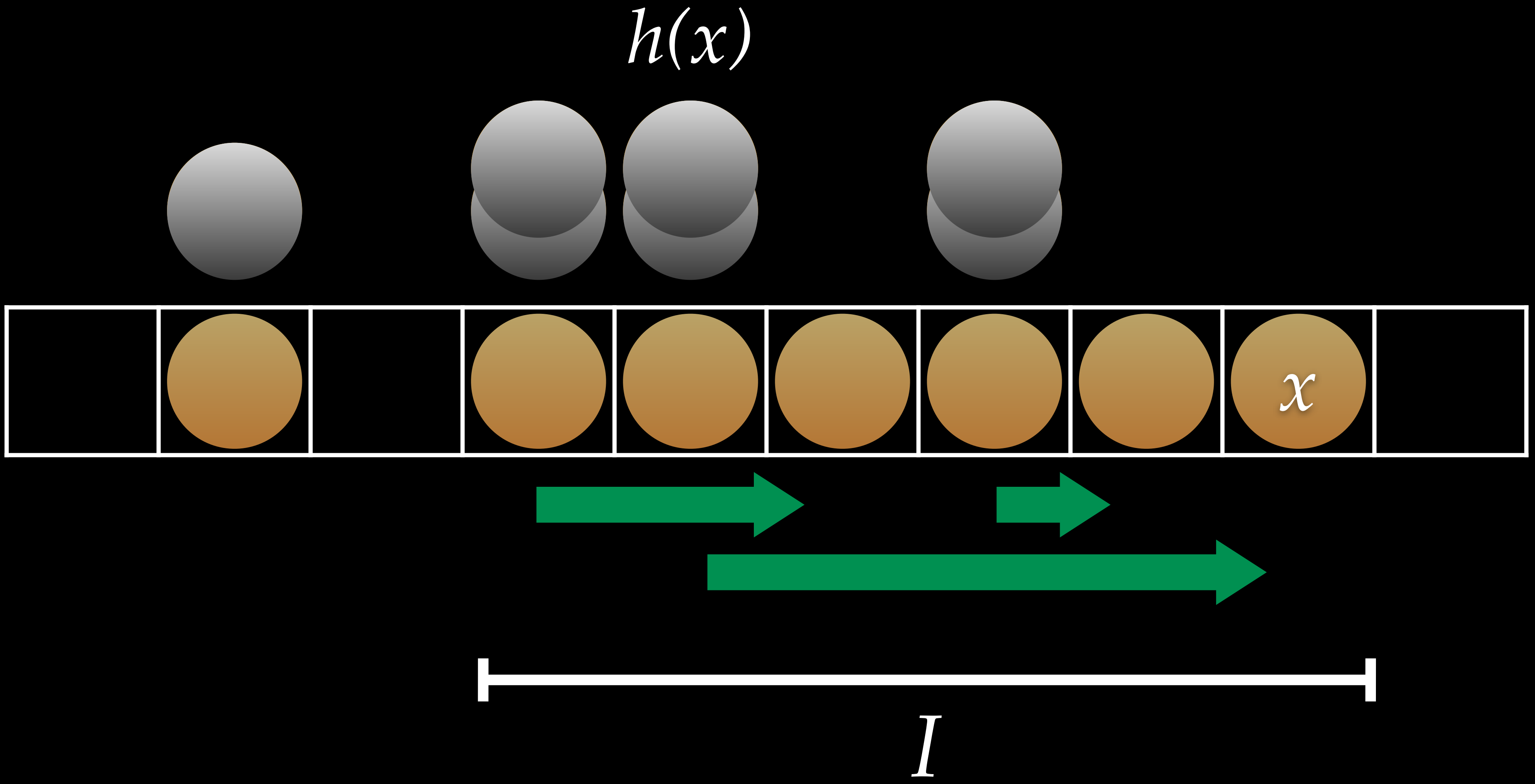


Notation: $L_I = |\{x \in S \mid h(x) \in I\}|$

Lemma. If insertion of a key x requires k probes, then there exists an interval I of length at least k such that $h(x) \in I$ and $L_I \geq |I|$.



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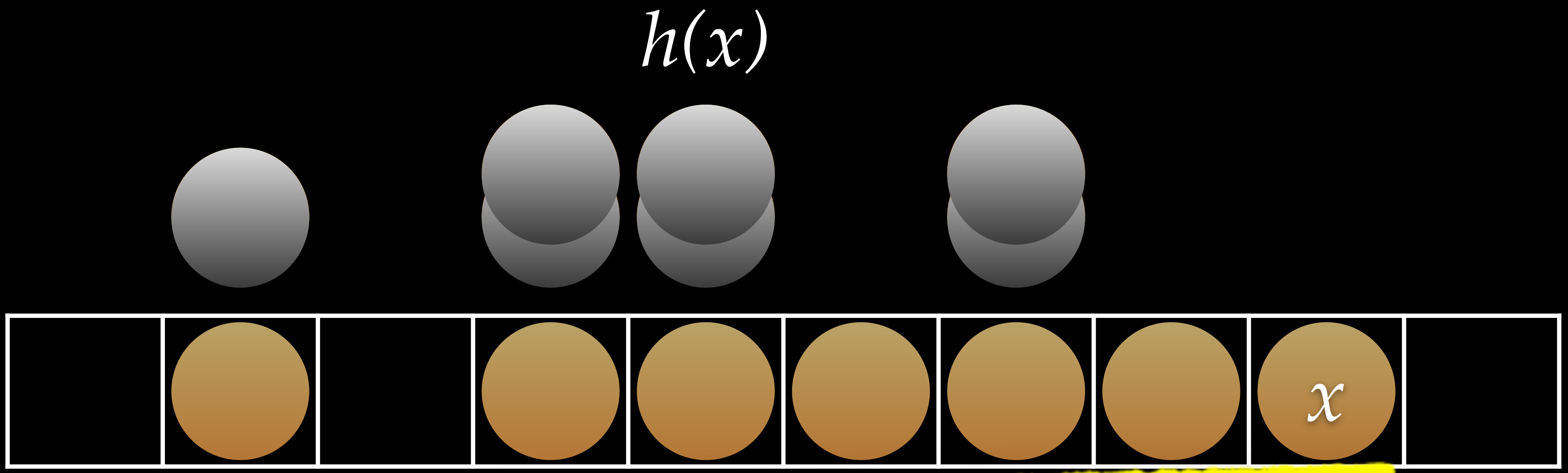

Insertion time is at most the number of "full" intervals around $h(x)$

How many “full” intervals?

- Assume that $r = 2n$, so we expect $L_I = |I|/2$. By Chernoff bounds:

$$\Pr[L_I > 2\mathbf{E}[L_I]] < (e/4)^{\mathbf{E}[L_I]}$$

Chernoff bounds are found in books on randomized algorithms or e.g. www.cs.uiuc.edu/~jeffe/teaching/algorithms/notes/11-chernoff.pdf

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- Expected number of full intervals around $h(x)$:

$$< \sum_{k=1}^{n} (e/4)^{-k/2} k = O(1)$$

- Assumes that values $h(x)$ are independent!

With 7-independence

- Fix a particular interval *I* containing *h*(*x*). Want to analyze prob. that L_I has |*I*| items.

- Define: $\ell(I) = \Pr[h(y) \in I] \leq |I|/2$
$$Y_x = \begin{cases} 1 - \ell(I), & \text{if } h(x) \in I \\ -\ell(I), & \text{otherwise} \end{cases} .$$

- Obs: $\sum_{x \in S} Y_x = L_I - \mathbf{E}[L_I] = L_I - n\ell(I)$

6th moment tail bound

$$\begin{aligned}\Pr[\sum_{x\in S} Y_x > |I|/2] &= \Pr[(\sum_{x\in S} Y_x)^6 > (|I|/2)^6] \\ &< \mathbf{E}[(\sum_{x\in S} Y_x)^6]/(|I|/2)^6 \\ &< 512/|I|^3\end{aligned}$$

- The first inequality is Markov's.
- The 2nd inequality requires that variables Y_x are 6-independent (and a calculation).

Concluding the argument

- Expected number of full intervals around *h*(*x*) is bounded by:

$$\sum_{k=1}^{n} \sum_{I \ni h(x), |I|=k} \Pr[L_I \geq |I|] \leq \sum_{k=1}^{n} k(512/k^3)$$

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Insertion time is at most the number of "full" intervals around $h(x)$

Tighter analysis: 5-independence works 4-independence does not

Some references

- Patrascu and Thorup: On the k-Independence Required by Linear Probing and Minwise Independence.
  http://people.csail.mit.edu/mip/papers/kwise-lb/kwise-lb.pdf (particularly section 1.1)
- Pagh, Pagh, and Ruzic: Linear probing with 5-wise independence
  http://www.itu.dk/people/pagh/papers/linear-sigest.pdf
- Thorup: String Hashing for Linear Probing
  https://www.siam.org/proceedings/soda/2009/SODA09_072_thorupm.pdf

Epilogue: Deterministic hashing

- Java string hashing (signed 32-bit arithmetic):
  $h(a_1a_2\ldots a_\mathrm{n}) = a_\mathrm{n} + 31\ h(a_1a_2\ldots a_\mathrm{n-1})$

- Collisions:
  - $h(Aa) = h(BB) = 2112$ (equivalent substrings)
  - $h(AaAa) = h(AaBB) = h(BBAa) = h(BBBB) = 2095104$
  - …

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- Recent heuristic hash functions, with focus on evaluation time: MurmurHash, CityHash, SipHash.

(Some) people are starting to care!

- Crosby & Wallach: *Denial of Service via Algorithmic Complexity Attacks*. Usenix Security ’03.
    - Follow-ups: Chaos Communication Congress ’11, ’12.

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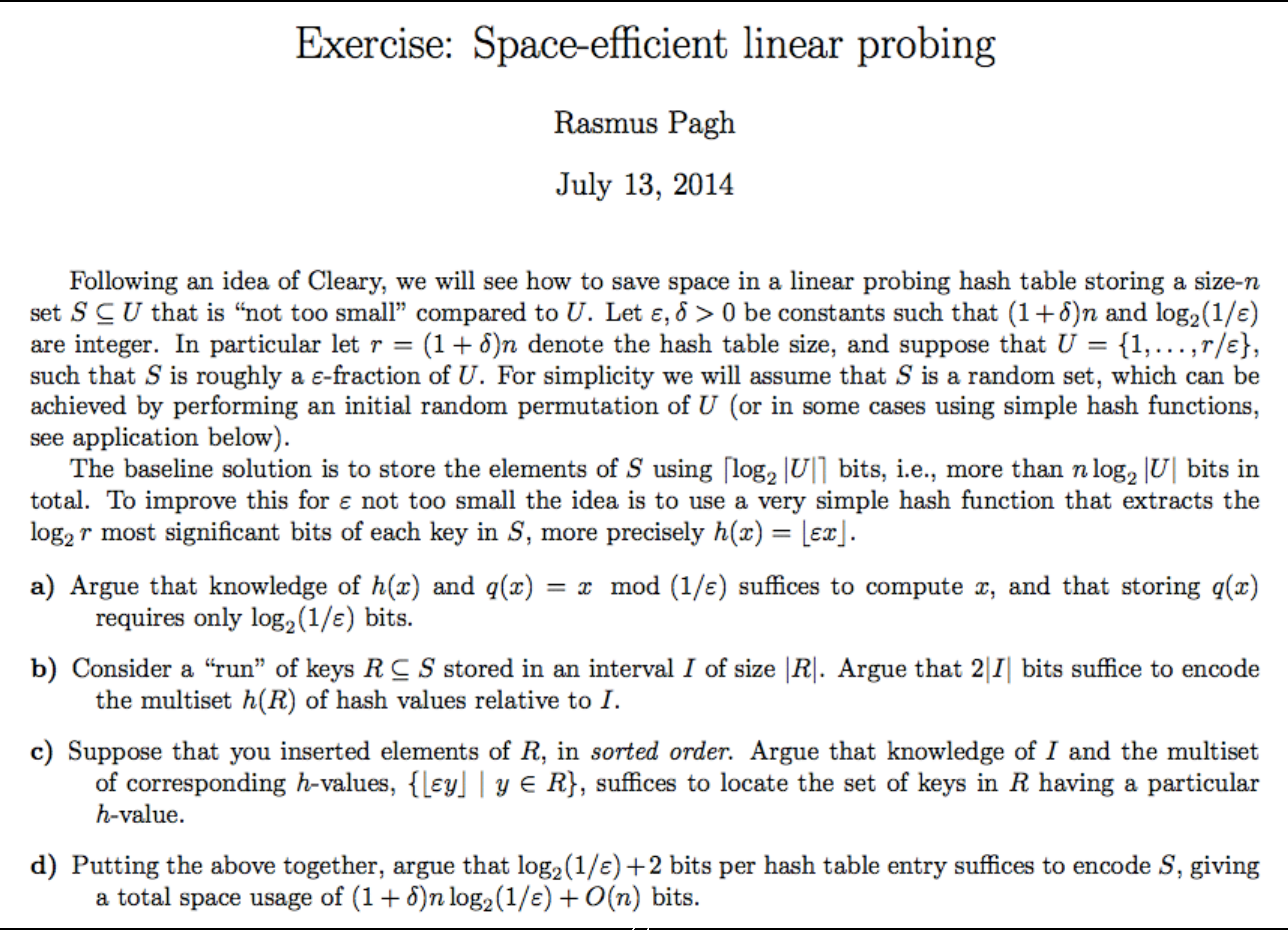
    - Follow-ups: Chaos Communication Congress ’11, ’12.
- Java, C++, C# libraries still use deterministic hashing.
    - Java falls back to BST for long hash chains!
- **NEW**: Ruby 1.9, Python 3.3, [Perl 5.18] now use *random hashing* [if deterministic hashing fails].



Exercise: Space-efficient linear probing

Rasmus Pagh

July 13, 2014

Following an idea of Cleary, we will see how to save space in a linear probing hash table storing a size-n set $S \subseteq U$ that is "not too small" compared to U. Let $\varepsilon, \delta > 0$ be constants such that $(1+\delta)n$ and $\log_2(1/\varepsilon)$ are integer. In particular let $r = (1+\delta)n$ denote the hash table size, and suppose that $U = \{1, \ldots, r/\varepsilon\}$, such that S is roughly a ε-fraction of U. For simplicity we will assume that S is a random set, which can be achieved by performing an initial random permutation of U (or in some cases using simple hash functions, see application below).

The baseline solution is to store the elements of S using $\lceil \log_2 |U| \rceil$ bits, i.e., more than $n \log_2 |U|$ bits in total. To improve this for ε not too small the idea is to use a very simple hash function that extracts the $\log_2 r$ most significant bits of each key in S, more precisely $h(x) = \lfloor \varepsilon x \rfloor$.

a) Argue that knowledge of $h(x)$ and $q(x) = x \mod (1/\varepsilon)$ suffices to compute x, and that storing $q(x)$ requires only $\log_2(1/\varepsilon)$ bits.

b) Consider a "run" of keys $R \subseteq S$ stored in an interval I of size $|R|$. Argue that $2|I|$ bits suffice to encode the multiset $h(R)$ of hash values relative to I.

c) Suppose that you inserted elements of R, in *sorted order*. Argue that knowledge of I and the multiset of corresponding h-values, $\{\lfloor \varepsilon y \rfloor \mid y \in R\}$, suffices to locate the set of keys in R having a particular h-value.

d) Putting the above together, argue that $\log_2(1/\varepsilon) + 2$ bits per hash table entry suffices to encode S, giving a total space usage of $(1+\delta)n \log_2(1/\varepsilon) + O(n)$ bits.