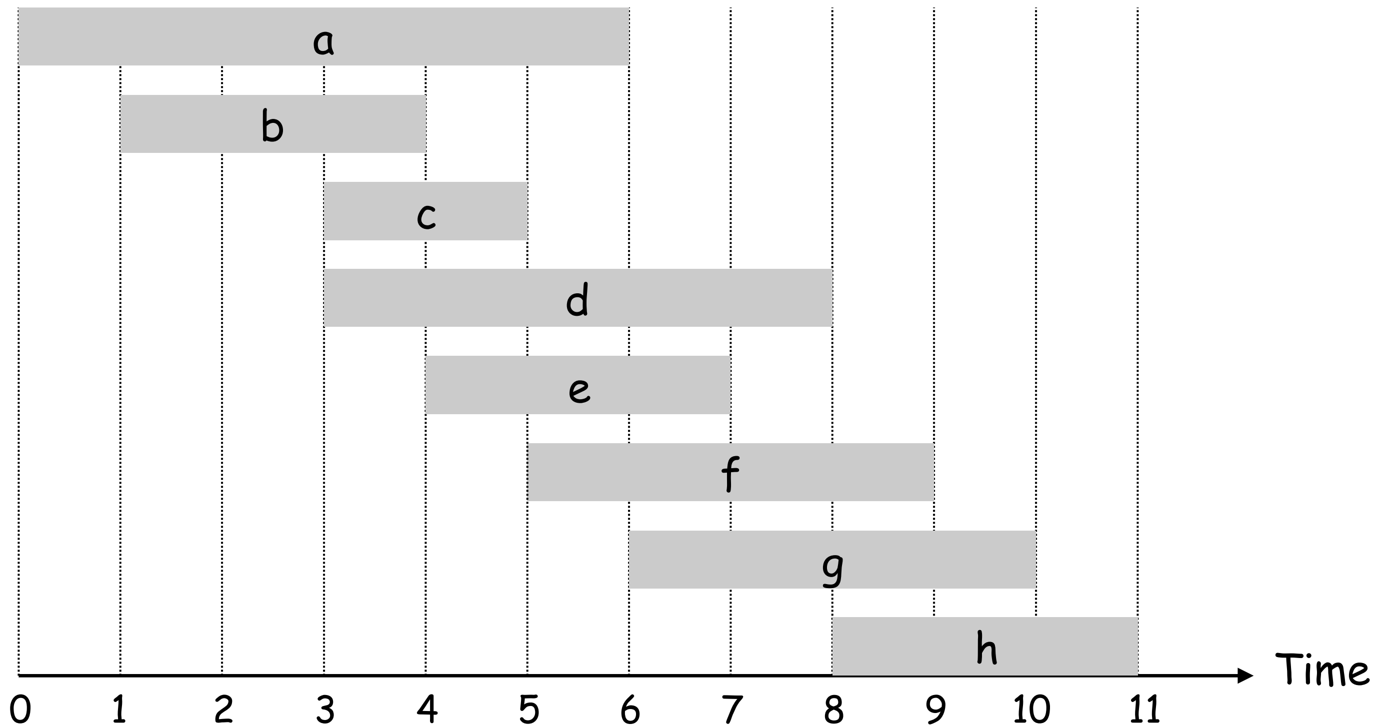
6.1 Weighted Interval Scheduling

Weighted Interval Scheduling

Weighted interval scheduling problem.

- Job j starts at s_j, finishes at f_j, and has weight or value v_j .
- Two jobs **compatible** if they don't overlap.
- Goal: find maximum **weight** subset of mutually compatible jobs.

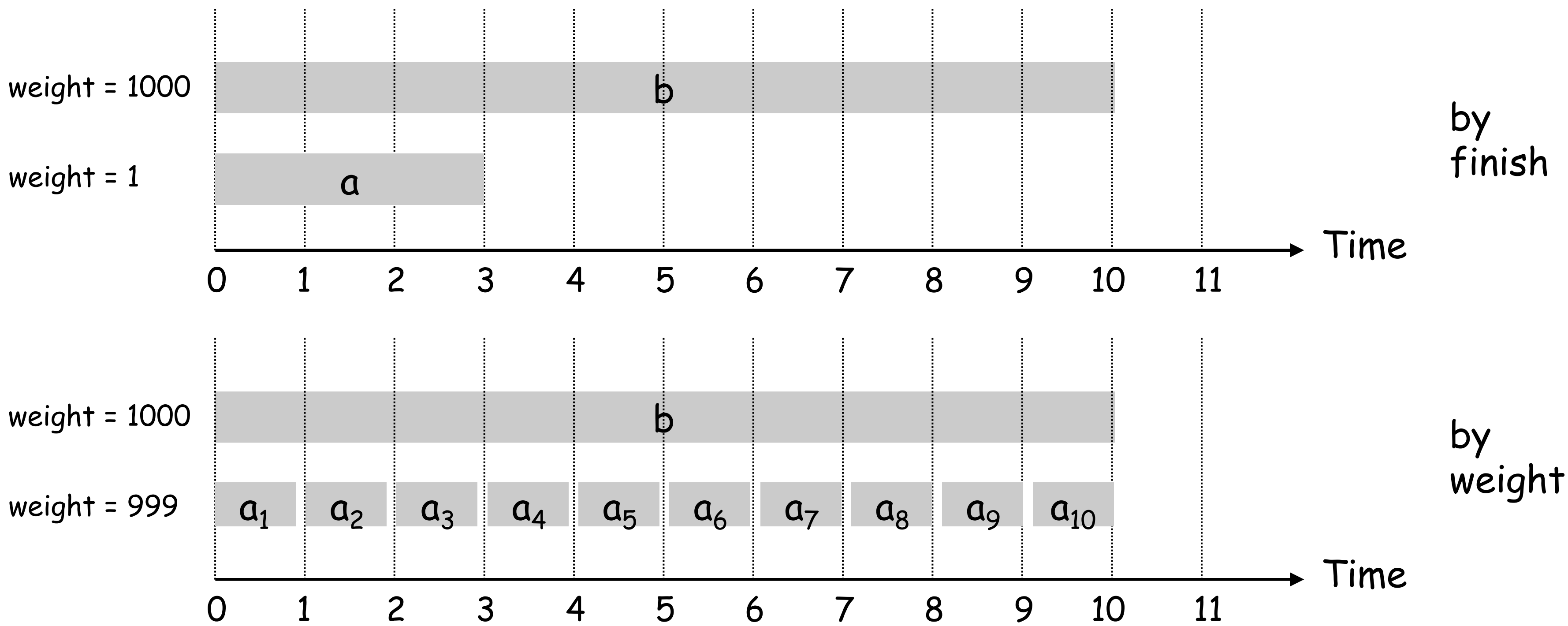


Unweighted Interval Scheduling Review

Recall. Greedy algorithm works if all weights are 1.

- Consider jobs in ascending order of finish time.
- Add job to subset if it is compatible with previously chosen jobs.

Observation. Greedy algorithm can fail spectacularly if arbitrary weights are allowed.

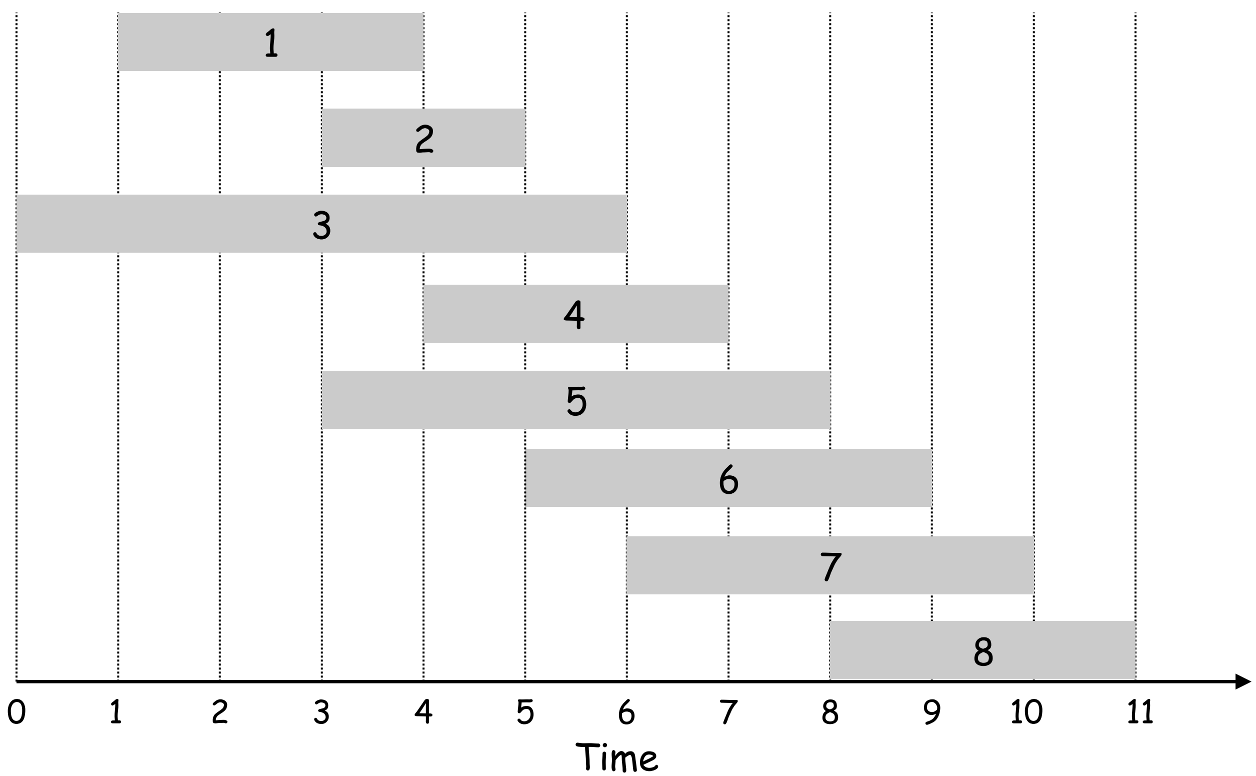


Weighted Interval Scheduling

Notation. Label jobs by finishing time: $f_1 \leq f_2 \leq \ldots \leq f_n$.

Def. $p(j)$ = largest index $i < j$ such that job i is compatible with j.

Ex: $p(8) = 5$, $p(7) = 3$, $p(2) = 0$.



j	p(j)
0	-
1	0
2	0
3	0
4	1
5	0
6	2
7	3
8	5

Dynamic Programming: Binary Choice

Notation. OPT(j) = value of optimal solution to the problem consisting of job requests 1, 2, ..., j.

- Case 1: Optimum selects job j.
  - can't use incompatible jobs { p(j) + 1, p(j) + 2, ..., j - 1 }
  - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., p(j)

optimal substructure

- Case 2: Optimum does not select job j.
  - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., j-1

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max \left\{ v_j + OPT(p(j)),\ OPT(j-1) \right\} & \text{otherwise} \end{cases}$$

Weighted Interval Scheduling: Brute Force

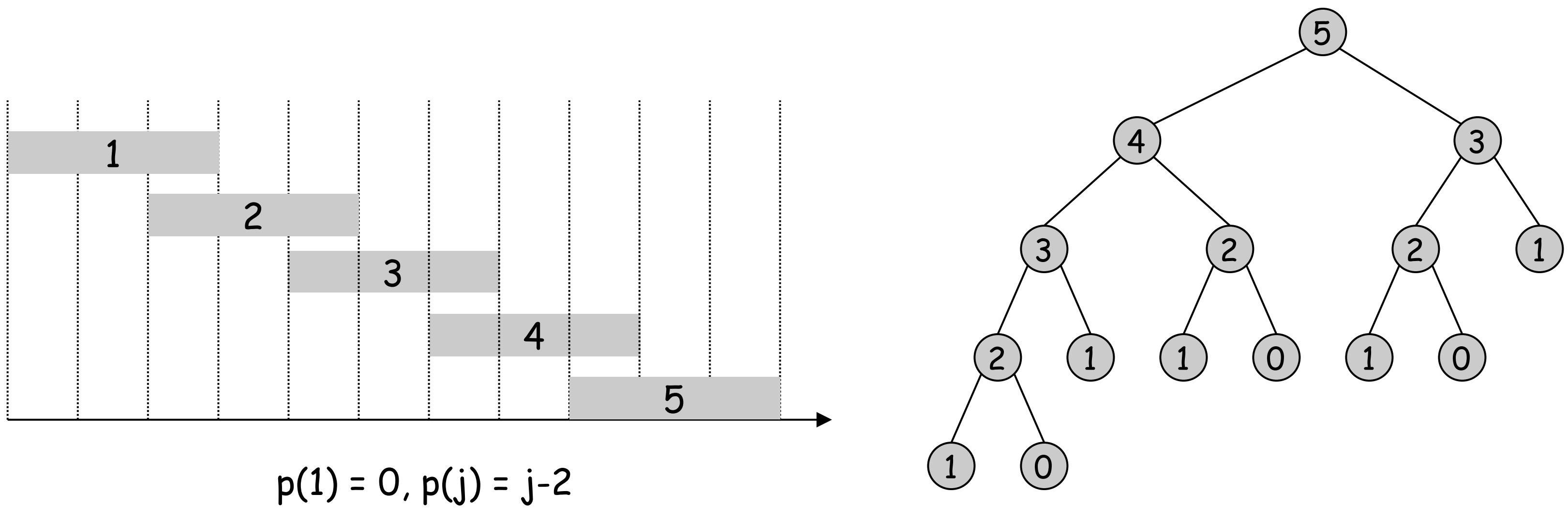
Brute force recursive algorithm.

```
Input: n, s_1,…,s_n , f_1,…,f_n , v_1,…,v_n

Sort jobs by finish times so that f_1 ≤ f_2 ≤ ... ≤ f_n.

Compute p(1), p(2), …, p(n)

Compute-Opt(j) {
   if (j = 0)
      return 0
   else
      return max(v_j + Compute-Opt(p(j)), Compute-Opt(j-1))
}
```

Weighted Interval Scheduling: Brute Force

Observation. Recursive algorithm fails spectacularly because of redundant sub-problems $\Rightarrow$ exponential algorithms.

Ex. Number of recursive calls for family of "layered" instances grows like Fibonacci sequence.



Weighted Interval Scheduling: Bottom-Up

Bottom-up dynamic programming. Unwind recursion.

```
Input: n, s_1,…,s_n , f_1,…,f_n , v_1,…,v_n

Sort jobs by finish times so that f_1 ≤ f_2 ≤ ... ≤ f_n.

Compute p(1), p(2), …, p(n)

Iterative-Compute-Opt {
   OPT[0] = 0
   for j = 1 to n
      OPT[j] = max(v_j + OPT[p(j)], OPT[j-1])
}

Output OPT[n]
```

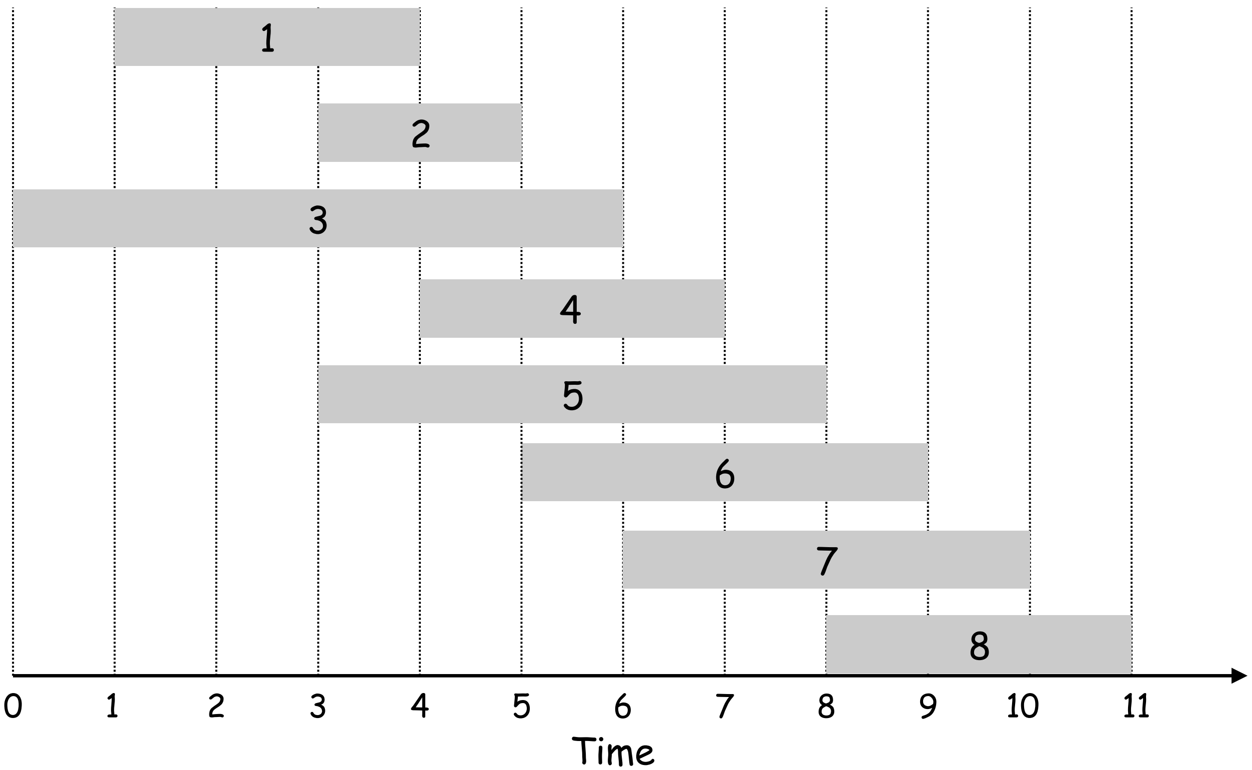
Claim: OPT[j] is value of optimal solution for jobs 1..j

Timing: Easy. Main loop is O(n); sorting is O(n log n)

Weighted Interval Scheduling

Notation. Label jobs by finishing time: $f_1 \le f_2 \le \ldots \le f_n$.
Def. p(j) = largest index i < j such that job i is compatible with j.

Ex: p(8) = 5, p(7) = 3, p(2) = 0.



j	vj	pj	optj
0	-	-	0
1		0	
2		0	
3		0	
4		1	
5		0	
6		2	
7		3	
8		5	

Weighted Interval Scheduling: Finding a Solution

Q. Dynamic programming algorithms computes optimal value. What if we want the solution itself?

A. Do some post-processing – "traceback"

```
Run M-Compute-Opt(n)
Run Find-Solution(n)

Find-Solution(j) {
   if (j = 0)
      output nothing
   else if (vj + OPT[p(j)] > OPT[j-1])
      print j
      Find-Solution(p(j))
   else
      Find-Solution(j-1)
}
```

the condition determining the max when computing OPT[ ] (refers to: else if (v_j + OPT[p(j)] > OPT[j-1]))

the relevant sub-problem (refers to: Find-Solution(p(j)) and Find-Solution(j-1))

- # of recursive calls $\leq n \Rightarrow O(n)$.

Sidebar: why does job ordering matter?

It's *Not* for the same reason as in the greedy algorithm for unweighted interval scheduling.

Instead, it's because it allows us to consider only a small number of subproblems (O(n)), vs the exponential number that seem to be needed if the jobs aren't ordered (seemingly, *any* of the 2^n possible subsets might be relevant)

Don't believe me? Think about the analogous problem for weighted *rectangles* instead of intervals… (I.e., pick max weight non-overlapping subset of a set of axis-parallel rectangles.) Same problem for circles also appears difficult.

6.4 Knapsack Problem

Knapsack Problem

Knapsack problem.

- Given n objects and a "knapsack."
- Item i weighs $w_i > 0$ kilograms and has value $v_i > 0$.
- Knapsack has capacity of W kilograms.
- Goal: fill knapsack so as to maximize total value.

Ex: { 3, 4 } has value 40.

W = 11

Item	Value	Weight	V/W
1	1	1	1
2	6	2	3
3	18	5	3.60
4	22	6	3.66
5	28	7	4

Greedy: repeatedly add item with maximum ratio v_i / w_i.

Ex: { 5, 2, 1 } achieves only value = 35 $\Rightarrow$ greedy not optimal.

[NB greedy *is* optimal for "fractional knapsack": take #5 + 4/6 of #4]

Dynamic Programming: False Start

Def. OPT(i) = max profit subset of items 1, …, i.

- Case 1: OPT does not select item i.
  - OPT selects best of { 1, 2, …, i-1 }
- Case 2: OPT selects item i.
  - accepting item i does not immediately imply that we will have to reject other items
  - without knowing what other items were selected before i, we don't even know if we have enough room for i

Conclusion. Need more sub-problems!

Dynamic Programming: Adding a New Variable

Def. OPT(i, w) = max profit subset of items 1, …, i with weight limit w.

- Case 1: OPT does not select item i.
  - OPT selects best of { 1, 2, …, i-1 } using weight limit w
- Case 2: OPT selects item i.
  - new weight limit = $w - w_i$
  - OPT selects best of { 1, 2, …, i–1 } using this new weight limit

$$OPT(i,w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1,w) & \text{if } w_i > w \\ \max\{\, OPT(i-1,w),\;\; v_i + OPT(i-1,w-w_i) \,\} & \text{otherwise} \end{cases}$$

Knapsack Problem: Bottom-Up

Knapsack. Fill up an n-by-W array.

```
Input: n, w1,…,wN, v1,…,vN

for w = 0 to W
   OPT[0, w] = 0

for i = 1 to n
   for w = 1 to W
      if (wi > w)
         OPT[i, w] = OPT[i-1, w]
      else
         OPT[i, w] = max {OPT[i-1,w],vi+OPT[i-1,w-wi]}

return OPT[n, W]
```

Knapsack Algorithm

W + 1 →

n + 1 ↓

	0	1	2	3	4	5	6	7	8	9	10	11
φ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40

OPT: { 4, 3 }
value = 22 + 18 = 40

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

```
if (w_i > w)
  OPT[i, w] = OPT[i-1, w]
else
  OPT[i, w] = max{OPT[i-1,w],v_i+OPT[i-1,w-w_i]}
```

Knapsack Problem: Running Time

Running time. $\Theta(n\ W)$.
- Not polynomial in input size!
- "Pseudo-polynomial."
- Knapsack is NP-hard. [Chapter 8]

Knapsack approximation algorithm. There exists a polynomial algorithm that produces a feasible solution that has value within 0.01% (or any other desired factor) of optimum. [Section 11.8]