CSE 521: Design & Analysis of Algorithms I

NP-completeness

Paul Beame

Computational Complexity

- Classify problems according to the amount of computational resources used by the best algorithms that solve them

- Recall:
  - worst-case running time of an algorithm
    - **max** # steps algorithm takes on any input of size **n**

Relative Complexity of Problems

- Want a notion that allows us to compare the complexity of problems
  - Want to be able to make statements of the form

    "If we could solve problem **B** in polynomial time then we can solve problem **A** in polynomial time"

    "Problem **B** is at least as hard as problem **A**"

Polynomial Time Reduction

- **A** $\leq_P$ **B** if there is an algorithm for **A** using a 'black box' (subroutine) that solves **B** that
  - Uses only a polynomial number of steps
  - Makes only a polynomial number of calls to a subroutine for **B**
- Thus, poly time algorithm for **B** implies poly time algorithm for **A**
  - Not only is the number of calls polynomial but the size of the inputs on which the calls are made is polynomial!
- If you can prove there is no fast algorithm for **A**, then that proves there is no fast algorithm for **B**

Why the name reduction?

- Weird: it maps an easier problem into a harder one

- Same sense as saying Maxwell reduced the problem of analyzing electricity & magnetism to solving partial differential equations
  - solving partial differential equations in general is a much harder problem than solving E&M problems

A geek joke

- An engineer
  - is placed in a kitchen with an empty kettle on the table and told to boil water; she fills the kettle with water, puts it on the stove, turns on the gas and boils water.
  - she is next confronted with a kettle full of water sitting on the counter and told to boil water; she puts it on the stove, turns on the gas and boils water.
- A mathematician
  - is placed in a kitchen with an empty kettle on the table and told to boil water; he fills the kettle with water, puts it on the stove, turns on the gas and boils water.
  - he is next confronted with a kettle full of water sitting on the counter and told to boil water: he empties the kettle in the sink, places the empty kettle on the table and says, "I've reduced this to an already solved problem".

A Special kind of Polynomial-Time Reduction

- We will always use a restricted form of polynomial-time reduction often called Karp or many-one reduction

- **A** $\leq_P^1$ **B** if and only if there is an algorithm for **A** given a black box solving **B** that on input **x**
  - Runs for polynomial time computing an input **f(x)**
  - Makes one call to the black box for **B**
  - Returns the answer that the black box gave

  We say that the function **f** is the reduction

Reductions by Simple Equivalence

- Show: Independent-Set $\leq_P$ Clique
- Independent-Set:
  - Given a graph **G=(V,E)** and an integer **k**, is there a subset **U** of **V** with **|U| ≥ k** such that no two vertices in **U** are joined by an edge?
- Clique:
  - Given a graph **G=(V,E)** and an integer **k**, is there a subset **U** of **V** with **|U| ≥ k** such that every pair of vertices in **U** is joined by an edge?

Independent-Set $\leq_P$ Clique

- Given **(G,k)** as input to Independent-Set where **G=(V,E)**
- Transform to **(G',k)** where **G'=(V,E')** has the same vertices as **G** but **E'** consists of **precisely** those edges that are not edges of **G**
- **U** is an independent set in **G**

⇔ **U** is a clique in **G'**

More Reductions

- Show: Independent Set $\leq_P$ Vertex-Cover
- Vertex-Cover:
  - Given an undirected graph **G=(V,E)** and an integer **k** is there a subset **W** of **V** of size ≤ **k** such that every edge of **G** has at least one endpoint in **W**? (i.e. **W** covers all edges of **G**)?
- Independent-Set:
  - Given a graph **G=(V,E)** and an integer **k**, is there a subset **U** of **V** with **|U|** ≥ **k** such that no two vertices in **U** are joined by an edge?

Reduction Idea

- Claim: In a graph **G=(V,E)**, **S** is an independent set iff **V-S** is a vertex cover
- Proof:
  - ⇒ Let **S** be an independent set in **G**
    - Then **S** contains at most one endpoint of each edge of **G**
    - At least one endpoint must be in **V-S**
    - **V-S** is a vertex cover
  - ⇐Let **W=V-S** be a vertex cover of **G**
    - Then **S** does not contain both endpoints of any edge (else **W** would miss that edge)
    - **S** is an independent set

Reduction

- Map (G,k) to (G,n-k)
  - Previous lemma proves correctness

- Clearly polynomial time

- We also get that
  - Vertex-Cover $\leq_P$ Independent Set

Reductions from a Special Case to a General Case

- Show: Vertex-Cover $\leq_P$ Set-Cover
- Vertex-Cover:
  - Given an undirected graph **G=(V,E)** and an integer **k** is there a subset **W** of **V** of size at most **k** such that every edge of **G** has at least one endpoint in **W**? (i.e. **W** covers all edges of **G**)?
- Set-Cover:
  - Given a set **U** of **n** elements, a collection $S_1,\ldots,S_m$ of subsets of **U**, and an integer **k**, does there exist a collection of at most **k** sets whose union is equal to **U**?

The Simple Reduction

- Transformation **f** maps **(G=(V,E),k)** to **$(U,S_1,\ldots,S_m,k')$**
  - **U←E**
  - For each vertex **$v \in V$** create a set **S_v** containing all edges that touch **v**
  - **k'←k**
- Reduction **f** is clearly polynomial-time to compute
- We need to prove that the resulting algorithm gives the right answer.

Proof of Correctness

- Two directions:
  - If the answer to Vertex-Cover on **(G,k)** is YES then the answer for Set-Cover on **f(G,k)** is YES
    - If a set **W** of **k** vertices covers all edges then the collection $\{S_v \mid v \in W\}$ of **k** sets covers all of **U**
  - If the answer to Set-Cover on **f(G,k)** is YES then the answer for Vertex-Cover on **(G,k)** is YES
    - If a subcollection $S_{v_1},\ldots,S_{v_k}$ covers all of **U** then the set $\{v_1,\ldots,v_k\}$ is a vertex cover in **G**.

Decision problems

- Computational complexity usually analyzed using decision problems
  - answer is just 1 or 0 (yes or no).

- Why?
  - much simpler to deal with
  - *deciding* whether **G** has a path from s to t, is certainly no harder than *finding* a path from s to t in G, so a *lower* bound on deciding is also a lower bound on finding
  - Less important, but if you have a good decider, you can often use it to get a good finder.

Polynomial time

- Define **P** (polynomial-time) to be
  - the set of all decision problems solvable by algorithms whose worst-case running time is bounded by some polynomial in the input size.

- Many decision problems are not known to be in **P**
  - e.g. decisionTSP:
    - Given a weighted graph **G** and an integer **k**, does there exist a tour that visits all vertices in **G** having total weight ≤ **k**?

Satisfiability

- Boolean variables $x_1,\ldots,x_n$
  - taking values in $\{0,1\}$. 0=false, 1=true
- Literals
  - x_i or $\neg x_i$ for $i=1,\ldots,n$
- Clause
  - a logical OR of one or more literals
  - e.g. $(x_1 \vee \neg x_3 \vee x_7 \vee x_{12})$
- CNF formula
  - a logical AND of a bunch of clauses
- k-CNF formula
  - All clauses have exactly k (distinct) variables

Satisfiability

- CNF formula example

  $(x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_4 \vee x_3) \wedge (x_2 \vee \neg x_1 \vee x_3)$

- If there is some assignment of 0's and 1's to the variables that makes it true then we say the formula is satisfiable
  - the one above is, the following isn't
  - $x_1 \wedge (\neg x_1 \vee x_2) \wedge (\neg x_2 \vee x_3) \wedge \neg x_3$
- 3-SAT: Given a CNF formula F with 3 variables per clause, is it satisfiable?

Common property of these problems

- There is a special piece of information, a **short certificate** or proof, that allows you to **efficiently verify** (in polynomial-time) that the YES answer is correct. This certificate might be very hard to find

- e.g.
  - **DecisionTSP**: the tour itself,
  - **Independent-Set**, **Clique**: the set **U**
  - **3-SAT**: an assignment that makes **F** true.

The complexity class NP

NP consists of all decision problems where

- You can **verify** the **YES** answers efficiently (in polynomial time) given a short (polynomial-size) **certificate/proof**

And

- **No certificate/proof** can fool your polynomial time verifier into saying **YES** for a **NO** instance

More Precise Definition of NP

- A decision problem is in **NP** iff there is a polynomial time procedure **verify**(.,.), and an integer k such that
  - for every input **x** to the problem that is a **YES** instance there is a certificate **t** with $|t| \leq |x|^k$ such that **verify**(**x**,**t**) = **YES**

  and
  - for every input **x** to the problem that is a **NO** instance there does **not** exist a certificate **t** with $|t| \leq |x|^k$ such that **verify**(**x**,**t**) = **YES**

Solving NP problems without certificates

- The only **obvious algorithm** for most of these problems is **brute force**:
  - try all possible certificates and check each one to see if it works.
  - *Exponential* time:
    - 2^n truth assignments for n variables
    - $n!$ possible TSP tours of n vertices
    - $\binom{n}{k}$ possible k element subsets of n vertices
    - etc.

P is contained in NP

- For a problem in **P** the **verify** procedure can be written to simply ignore its certificate

- Note: Saying that a problem is an **NP** problem means that it is easy to check solutions. It does NOT mean that the problem is hard.

Problems in NP that seem hard

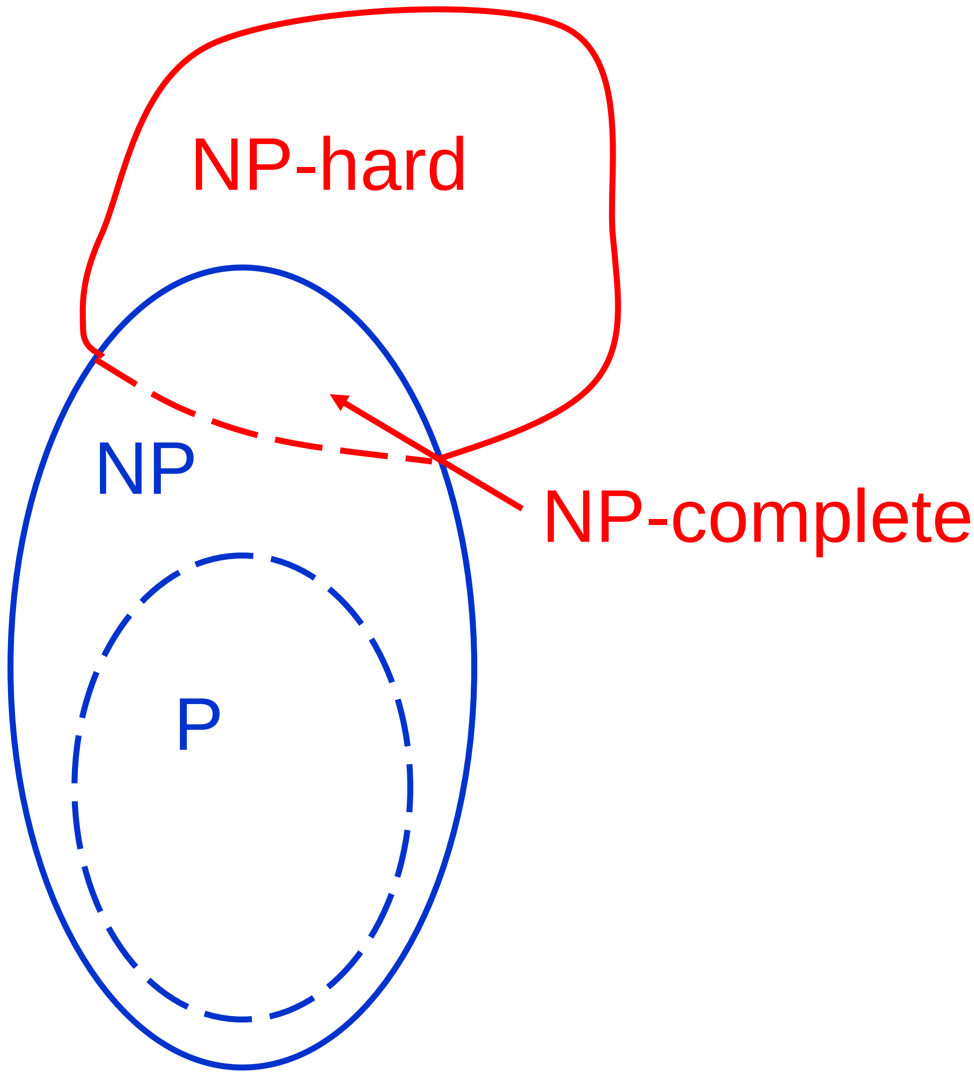
- Some examples in **NP**
  - 3-SAT
  - Independent-Set
  - Clique
  - Vertex Cover
- All hard to solve; certificates seem to help on all
- Fast solution to *any* gives fast solution to all of them

NP-hardness & NP-completeness

- Alternative approach to proving problems not in **P**
  - show that they are at least as hard as any problem in **NP**

- Rough definition:
  - A problem is NP-hard iff it is at least as hard as any problem in **NP**
  - A problem is NP-complete iff it is both
    - **NP**-hard
    - in **NP**

P and NP



NP-hardness & NP-completeness

- Definition: A problem **B** is NP-hard iff every problem **A**$\in$**NP** satisfies **A** $\leq_P$**B**
- Definition: A problem **B** is NP-complete iff **A** is NP-hard and **A** $\in$**NP**
- Even though we seem to have lots of hard problems in **NP** it is not obvious that such super-hard problems even exist!

Cook-Levin Theorem

- Theorem (Cook 1971, Levin 1973):

  3-SAT is **NP**-complete

- Recall
  - CNF formula
    - $(x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_4 \vee x_3) \wedge (x_2 \vee \neg x_1 \vee x_3)$
  - If there is some assignment of **0**'s and **1**'s to the variables that makes it true then we say the formula is satisfiable
  - **3-SAT:** Given a 3-CNF formula F, is it satisfiable?

Implications of Cook-Levin Theorem?

- There is at least one interesting problem in **NP** that captures the hardest **NP** problems

- Is that such a big deal?

- YES!
  - There are lots of other problems that can be solved if we had a polynomial-time algorithm for **3-SAT**
  - Many of these problems are exactly as hard as **3-SAT**

A useful property of polynomial-time reductions

- Theorem: If $A \leq_P B$ and $B \leq_P C$ then $A \leq_P C$
- Proof idea: (Using $\leq_P^1$)
  - Compose the reduction **f** from **A** to **B** with the reduction **g** from **B** to **C** to get a new reduction **h(x)=g(f(x))** from **A** to **C**.
    - Only work is to show the time bound since the reduction **f** may increase the input size for the reduction **g**
    - Uses the fact that the composition of two polynomials is also a polynomial
  - The general case is similar

Cook-Levin Theorem & Implications

- Theorem (Cook 1971, Levin 1973):
  3-SAT is **NP**-complete

  For proof see CSE 531

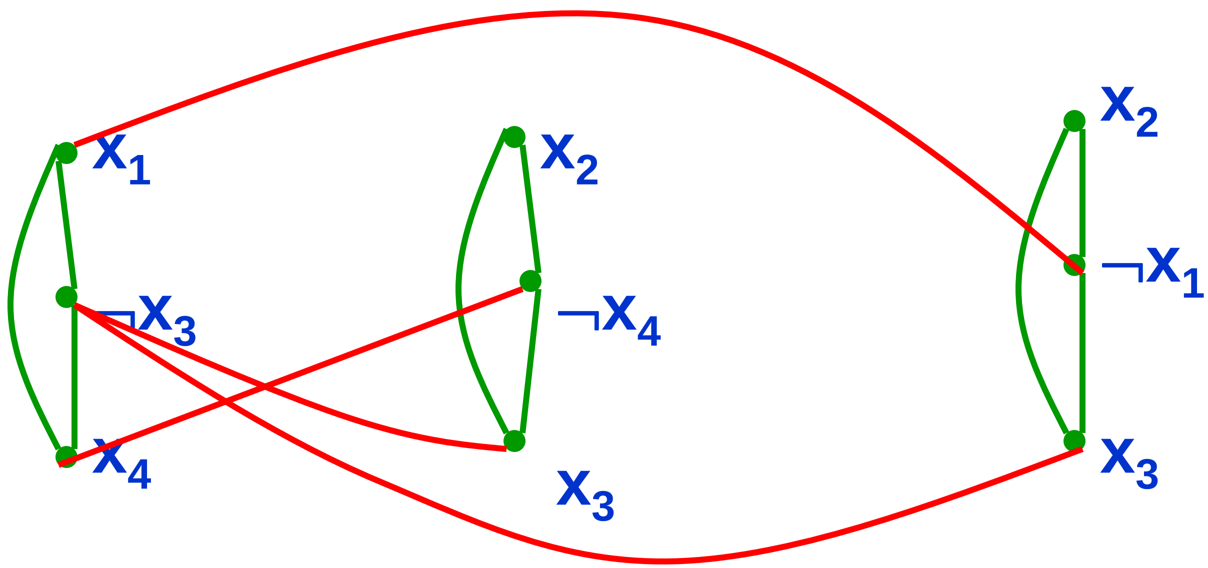
- Corollary: **B** is **NP**-hard $\Leftrightarrow$ **3-SAT** $\leq_P$ **B**
  - (or **A** $\leq_P$ **B** for any **NP**-complete problem **A**)
- Proof:
  - If **B** is **NP**-hard then every problem in **NP** polynomial-time reduces to **B**, in particular **3-SAT** does since it is in **NP**
  - For any problem **A** in **NP**, **A** $\leq_P$ **3-SAT** and so if **3-SAT** $\leq_P$ **B** we have **A** $\leq_P$ **B**.
    - therefore **B** is **NP**-hard if **3-SAT** $\leq_P$ **B**

Another NP-complete problem: 3-SAT $\leq_P$ Independent-Set

- A Tricky Reduction:
  - mapping CNF formula **F** to a pair **(G,k)**
  - Let **m** be the number of clauses of **F**
  - Create a vertex in **G** for each literal in **F**
  - Join two vertices **u**, **v** in **G** by an edge iff
    - **u** and **v** correspond to literals in the same clause of **F**, (green edges) or
    - **u** and **v** correspond to literals **x** and **¬x** (or vice versa) for some variable **x**. (red edges).
  - Set **k=m**
  - Clearly polynomial time

3-SAT $\leq_P$ Independent-Set

F: $(x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_4 \vee x_3) \wedge (x_2 \vee \neg x_1 \vee x_3)$



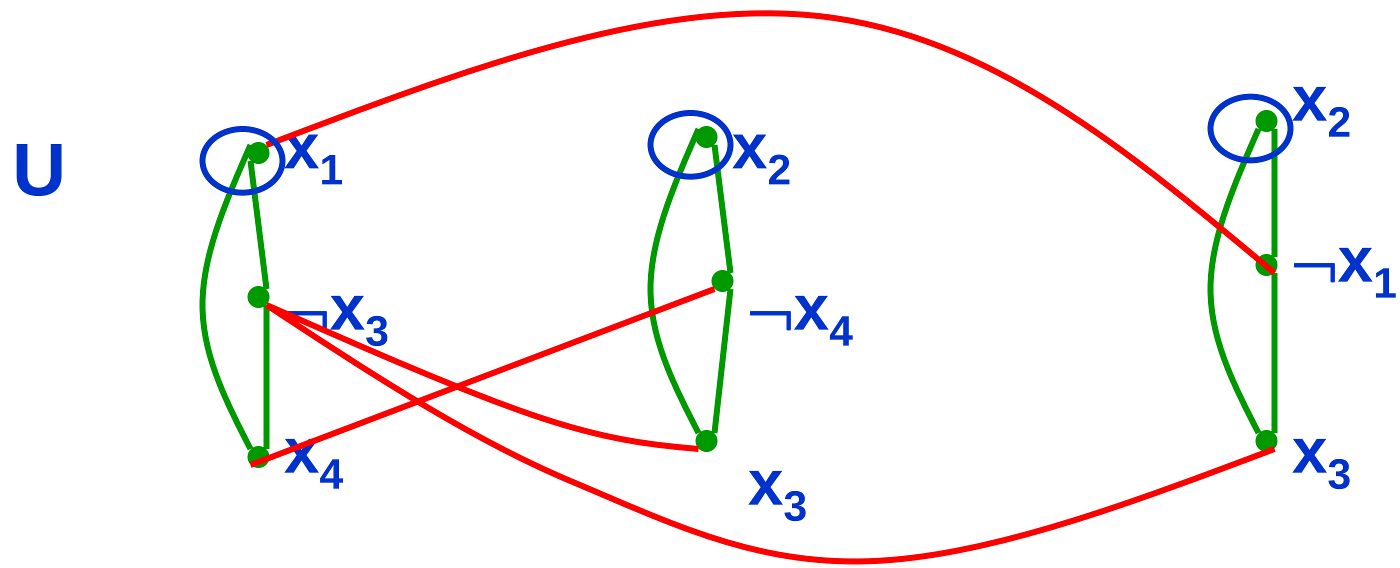
3-SAT $\leq_P$ Independent-Set

- Correctness:
  - If **F** is **satisfiable** then there is some assignment that satisfies at least one literal in each clause.
  - Consider the set **U** in **G** corresponding to the **first satisfied literal in each clause**.
    - **|U|=m**
    - Since **U** has only one vertex per clause, no two vertices in **U** are joined by green edges
    - Since a truth assignment never satisfies both **x** and **¬x**, **U** doesn't contain vertices labeled both **x** and **¬x** and so no vertices in **U** are joined by red edges
    - Therefore **G** has an independent set, **U**, of size at least **m**
  - Therefore (**G**,**m**) is a **YES** for independent set.

3-SAT $\leq_P$ Independent-Set

$$\begin{matrix} & 1 & 0 & 1 & & 1 & 0 & 1 & & 1 & 0 & 1 \end{matrix}$$

F: $(x_1 \lor \neg x_3 \lor x_4) \land (x_2 \lor \neg x_4 \lor x_3) \land (x_2 \lor \neg x_1 \lor x_3)$



Given assignment $x_1=x_2=x_3=x_4=1$,

U is as circled

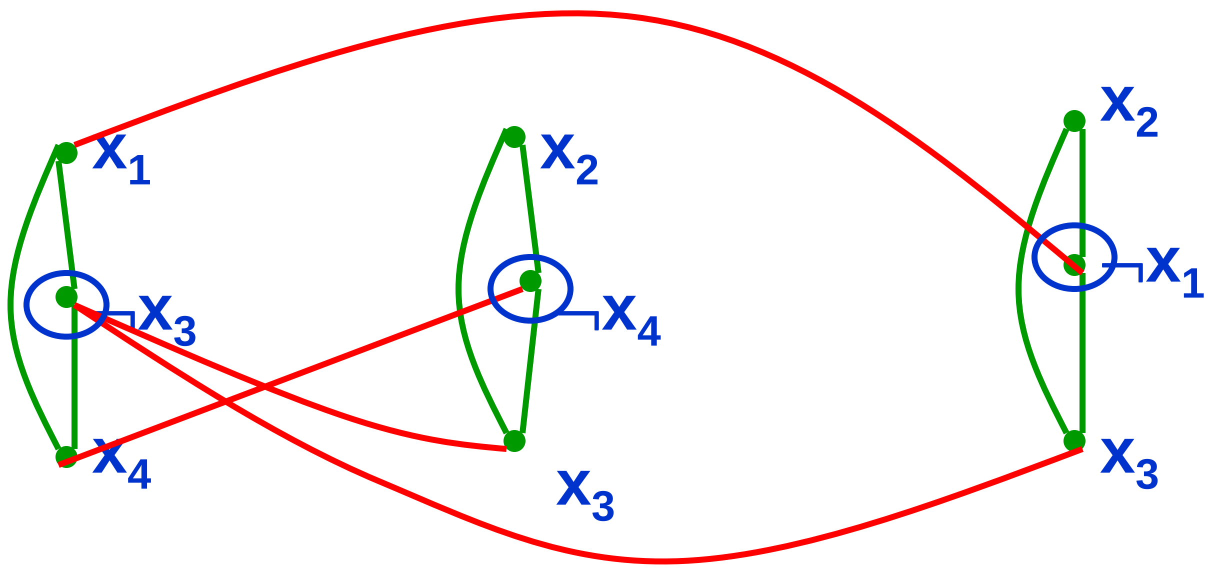
3-SAT $\leq_P$ Independent-Set

- Correctness continued:
  - If (G,m) is a **YES** for Independent-Set then there is a set **U** of **m** vertices in **G** containing no edge.
    - Therefore **U** has precisely one vertex per clause because of the green edges in **G**.
    - Because of the red edges in **G**, **U** does not contain vertices labeled both **x** and **¬x**
    - Build a truth assignment **A** that makes all literals labeling vertices in **U** true and for any variable not labeling a vertex in **U**, assigns its truth value arbitrarily.
    - By construction, **A** satisfies **F**
  - Therefore **F** is a **YES** for 3-SAT.

3-SAT $\leq_P$ Independent-Set

0 1 0 ? 1 0 ? 1 0

F: $(x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_4 \vee x_3) \wedge (x_2 \vee \neg x_1 \vee x_3)$



Given **U**, satisfying assignment is $x_1=x_3=x_4=0$, $x_2=0$ or 1

Independent-Set is NP-complete

- We just showed that **Independent-Set** is NP-hard and we already knew **Independent-Set** is in **NP**.

- Corollary: **Clique** is **NP**-complete
  - We showed already that **Independent-Set** $\leq_P$ **Clique** and **Clique** is in **NP**.

Problems we already know are NP-complete

- 3-SAT
- Independent-Set
- Clique
- Vertex-Cover
- Set-Cover

- 10,000’s of practical problems that are NP-complete, e.g. scheduling, optimal VLSI layout etc.

Steps to Proving Problem B is NP-complete

- Show **B** is **NP**-hard:
  - State:"Reduction is from **NP-**hard Problem **A**"
  - Show what the map **f** is
  - Argue that **f** is polynomial time
  - Argue correctness: two directions Yes for **A** implies Yes for **B** and vice versa.
- Show **B** is in **NP**
  - State what certificate is and why it works
  - Argue that it is polynomial-time to check.

Some other NP-complete examples you should know

- **Hamiltonian-Cycle** Given a directed graph G is there a cycle in G that visits each vertex in G exactly once?
- **Hamiltonian-Path** Given a directed graph G is there a path in G that visits each vertex in G exactly once?
  - Both are also **NP**-complete when G is an undirected graph
- Note that deciding the similar questions for **Eulerian-Cycle** and **Eulerian-Path** (which require that each edge be visited exactly once rather than each vertex) can be done in polynomial time.
  - How?

Travelling-Salesman Problem (TSP)

- Given a set of **n** cities $v_1,\ldots,v_n$ and distances between each pair of cities $d(v_i,v_j)$ what is the shortest tour that visits all the cities?
  - Not a decision problem

- **DecisionTSP**:
  - Given a set of distances given by **d** for each pair of cities in $v_1,\ldots,v_n$ and an integer **D**, does there exist a tour that visits all cities having total weight at most **D**?

Hamiltonian-Cycle $\leq_P$ DecisionTSP

- Define the reduction
  - Vertices **V** of **G=(V,E)** become cities
  - Set $d(v_i,v_j)$ to **1** if $(v_i,v_j)\in E$
    2 if not
  - Set $D=|V|$

- Claim: There is a Hamiltonian cycle in **G** iff there is a tour of length $|V|$

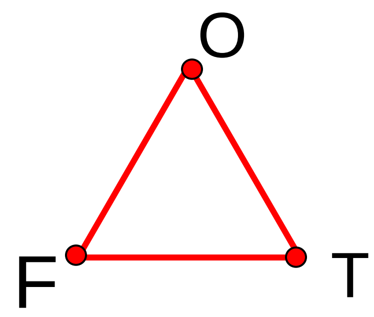
Graph Colorability

- Defn: Given a graph G=(V,E), and an integer k, a k-coloring of G is
  - an assignment of up to k different colors to the vertices of G so that the endpoints of each edge have different colors.
- 3-Color: Given a graph G=(V,E), does G have a 3-coloring?
- Claim: 3-Color is NP-complete
- Proof: 3-Color is in NP:
  - Certificate is an assignment of red,green,blue to the vertices of G
  - Easy to check that each edge is colored correctly

3-SAT $\leq_P$ 3-Color

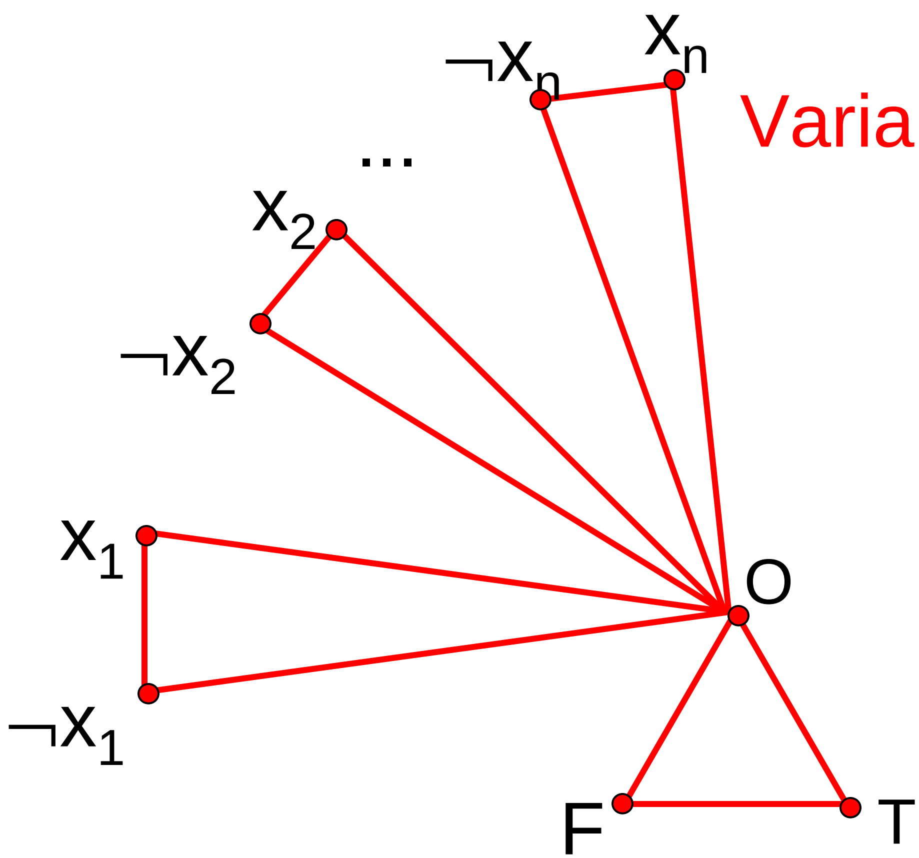
- Reduction:
  - We want to map a 3-CNF formula **F** to a graph **G** so that
    - **G** is 3-colorable iff **F** is satisfiable

3-SAT $\leq_P$ 3-Color



Base Triangle

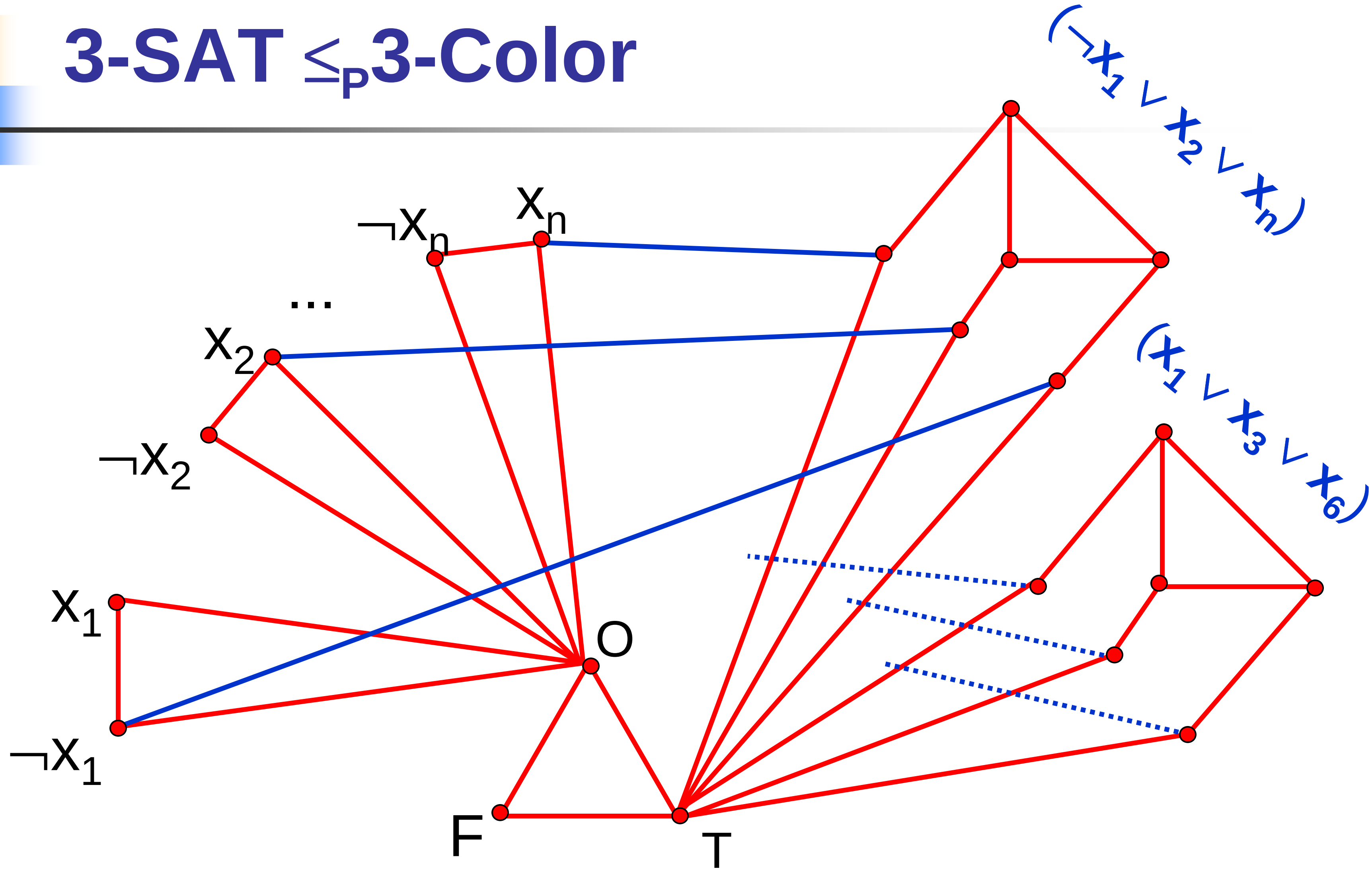
3-SAT $\leq_P$ 3-Color



Variable Part:

in 3-coloring, variable colors correspond to some truth assignment (same color as T or F)

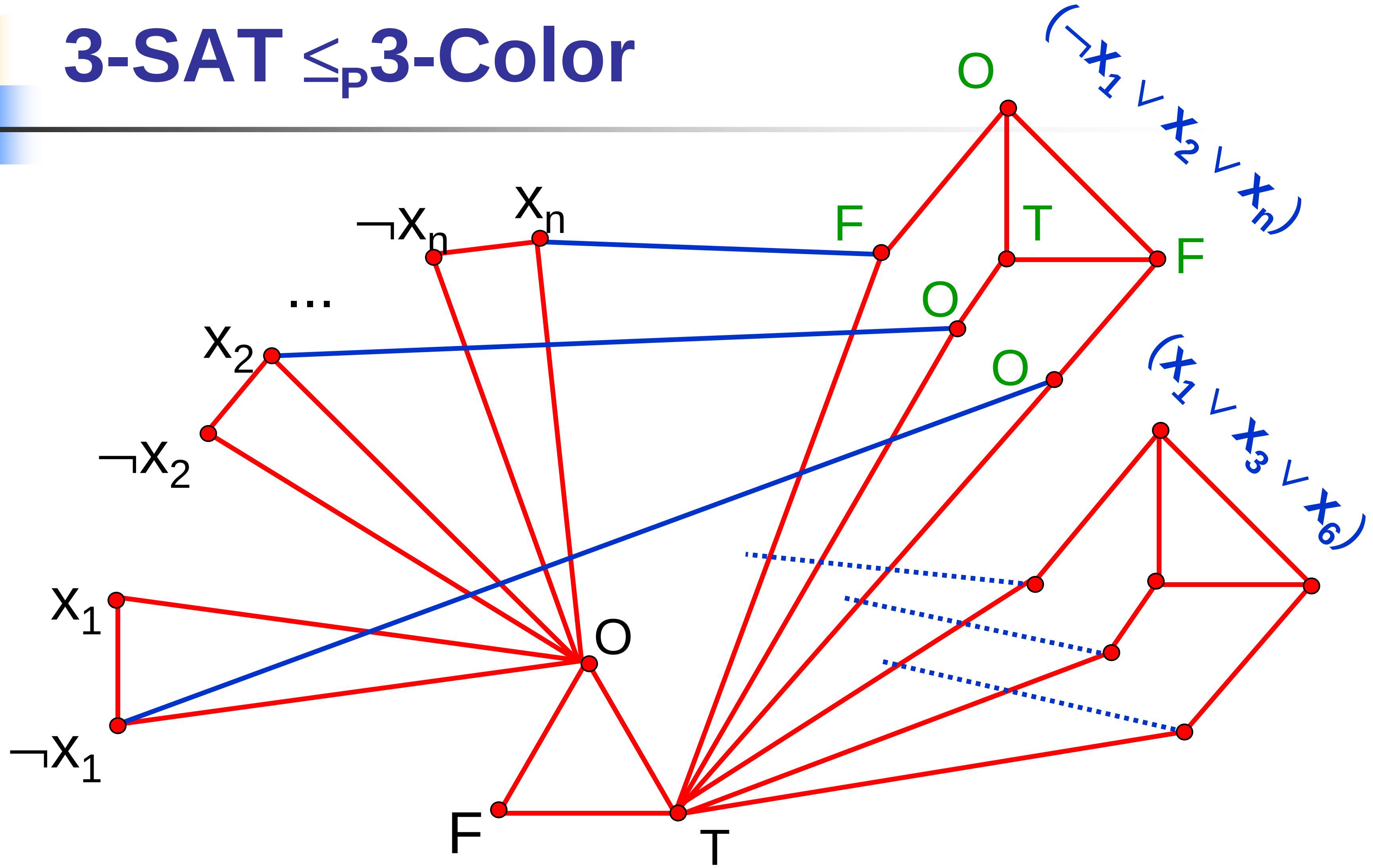
3-SAT $\leq_P$ 3-Color



Clause Part:

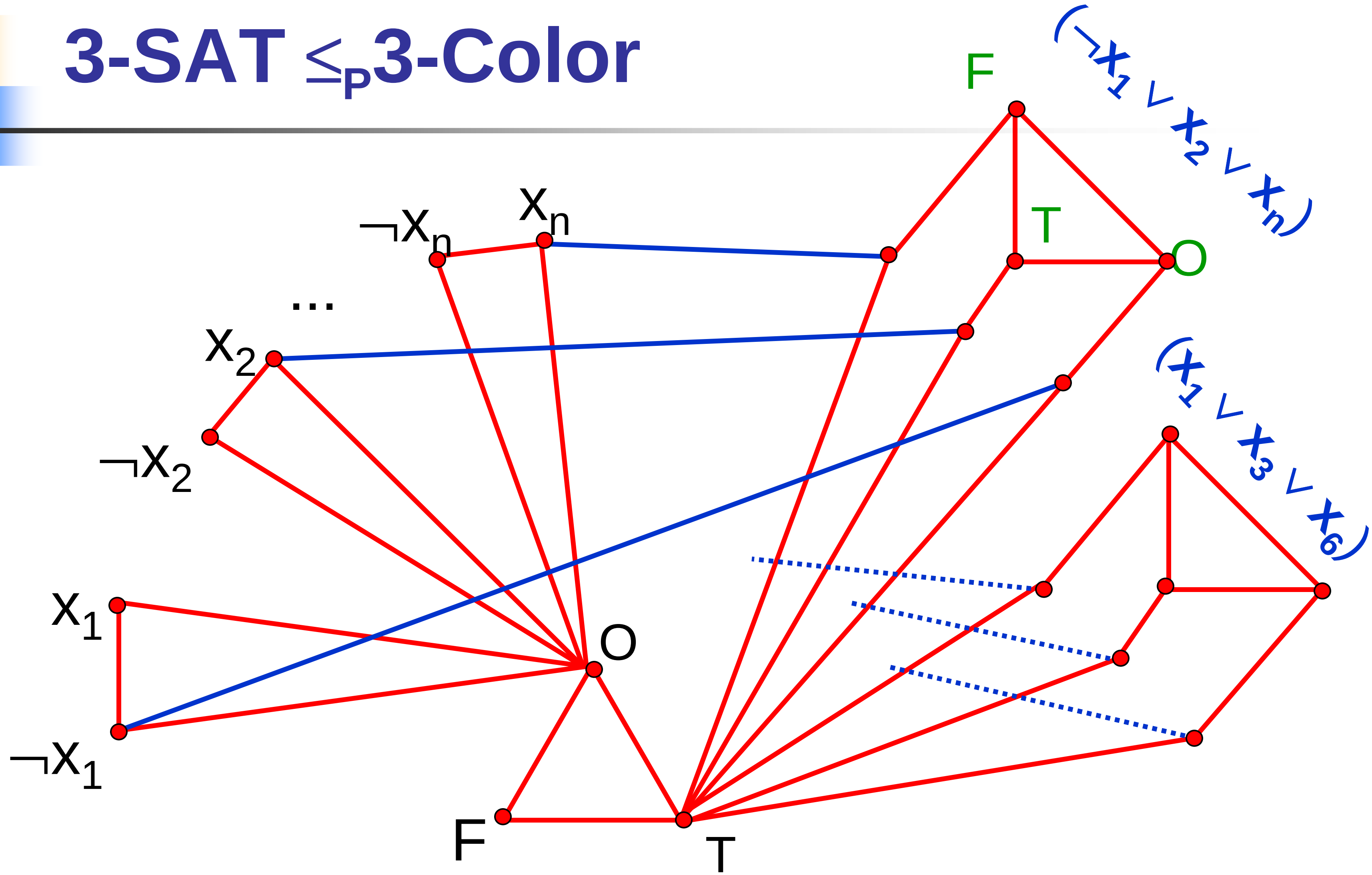
Add one 6 vertex gadget per clause connecting its 'outer vertices' to the literals in the clause

3-SAT $\leq_P$ 3-Color



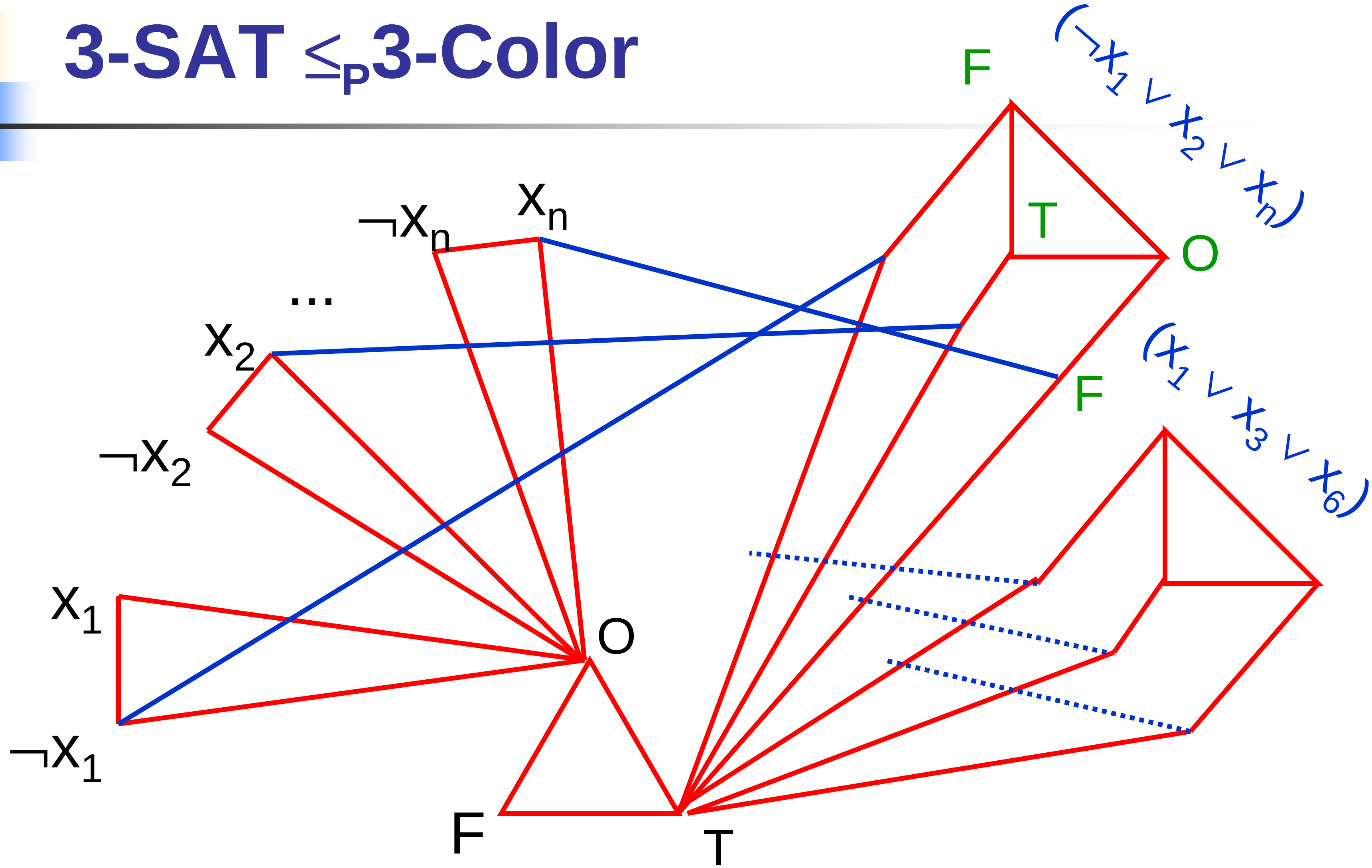
Any truth assignment satisfying the formula can be extended to a 3-coloring of the graph

3-SAT $\leq_P$ 3-Color



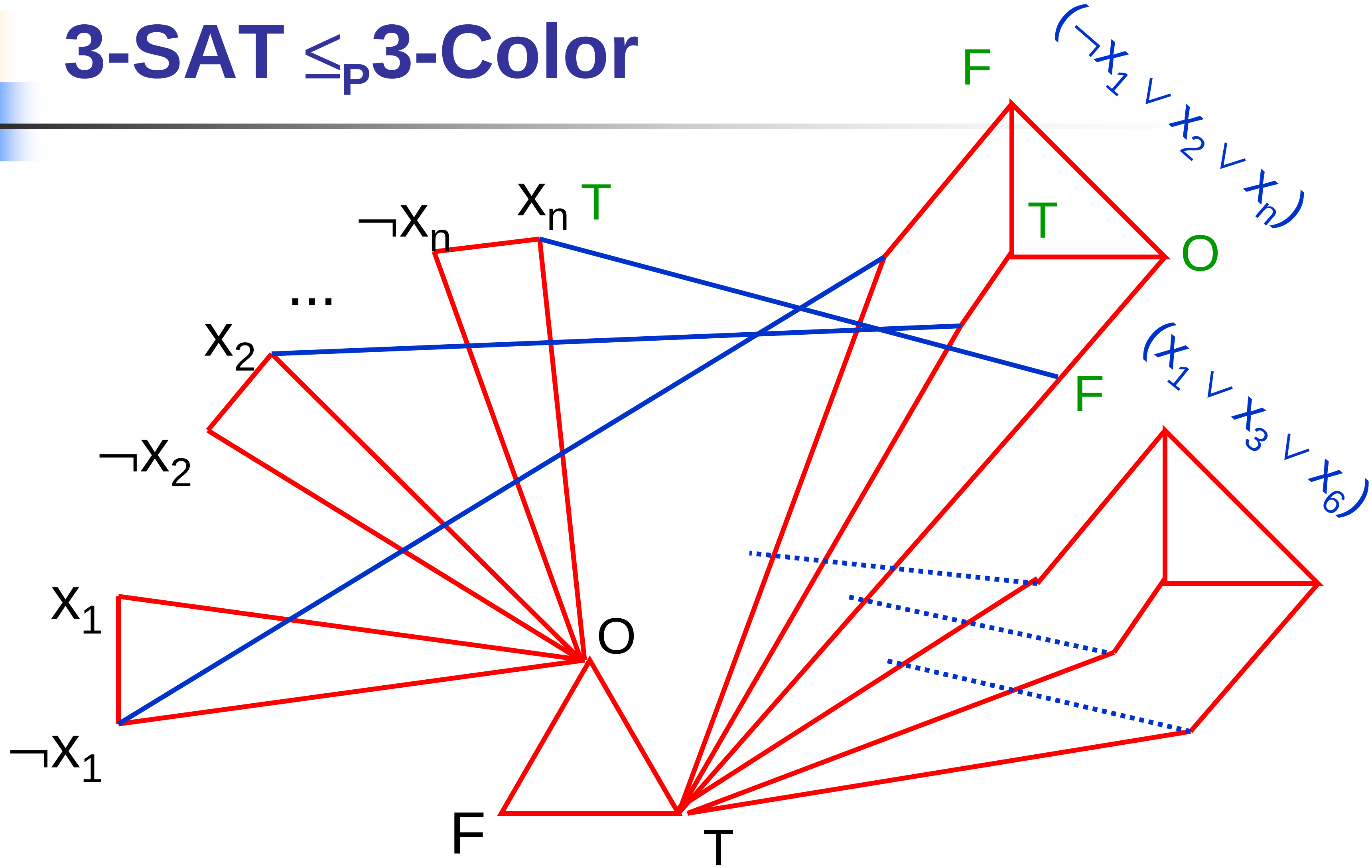
Any 3-coloring of the graph colors each gadget triangle using each color

3-SAT $\leq_P$ 3-Color



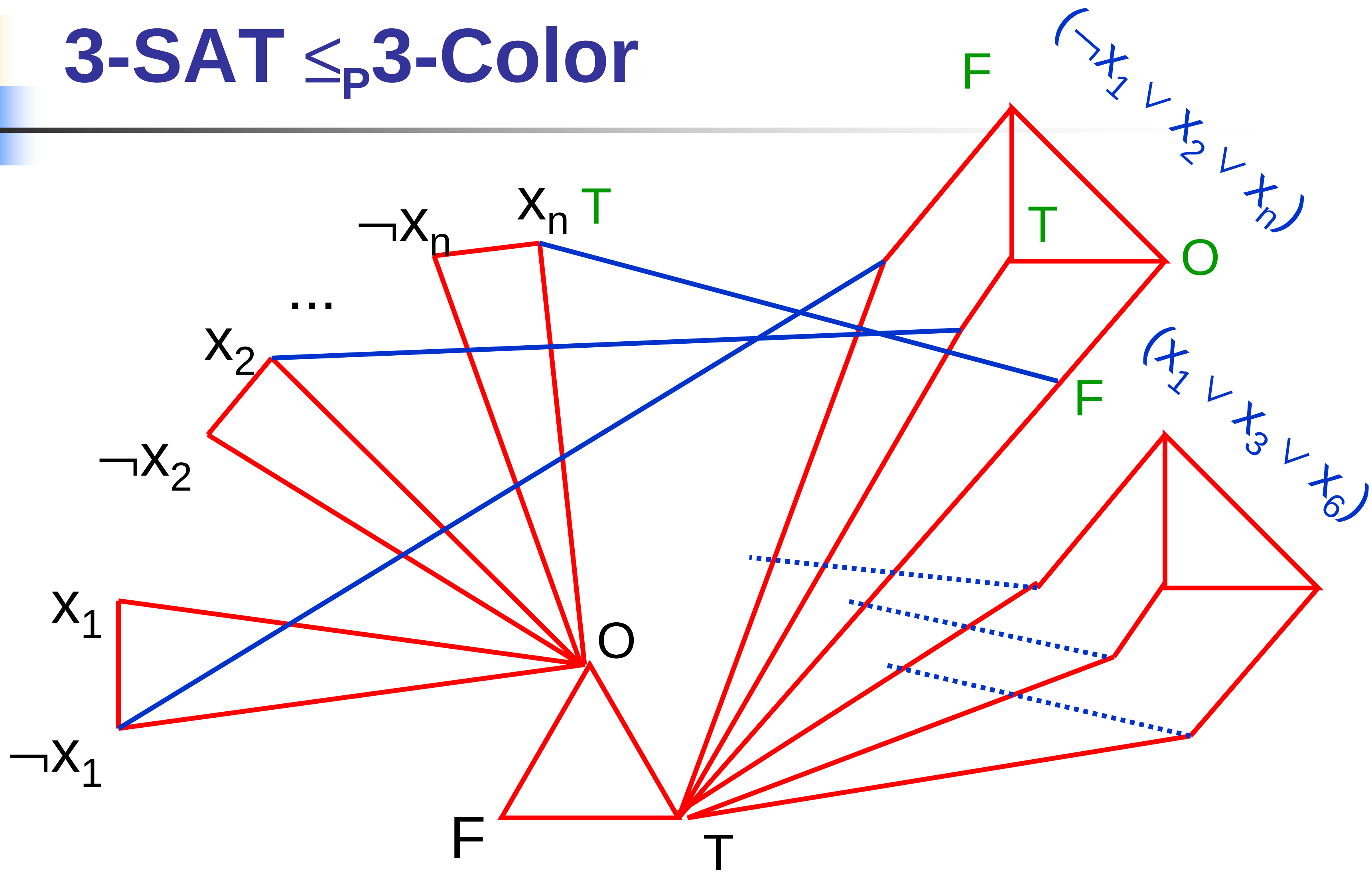
Any 3-coloring of the graph has an F opposite the O color in the triangle of each gadget

3-SAT $\leq_P$ 3-Color



Any 3-coloring of the graph has T at the other end of the blue edge connected to the F

3-SAT $\leq_P$ 3-Color



Any 3-coloring of the graph yields a satisfying assignment to the formula

More NP-completeness

- Subset-Sum problem
  - Given **n** integers $w_1,\ldots,w_n$ and integer **W**
  - Is there a subset of the **n** input integers that adds up to exactly **W**?

- **O(nW)** solution from dynamic programming but if **W** and each w_i can be **n** bits long then this is exponential time

3-SAT $\leq_P$ Subset-Sum

- Given a 3-CNF formula with **m** clauses and **n** variables
- Will create 2**m**+2**n** numbers that are **m+n** digits long
  - Two numbers for each variable x_i
    - t_i and f_i (corresponding to x_i being true or x_i being false)
  - Two extra numbers for each clause
    - u_j and v_j (filler variables to handle number of false literals in clause C_j)

3-SAT $\leq_P$ Subset-Sum

$C_3 = (x_1 \vee \neg x_2 \vee x_5)$

	i						j					
	1	2	3	4	…	n	1	2	3	4	…	m
t_1	1	0	0	0	…	0	0	0	1	0	…	1
f_1	1	0	0	0	…	0	1	0	0	1	…	0
t_2	0	1	0	0	…	0	0	1	0	0	…	1
f_2	0	1	0	0	…	0	0	0	1	1	…	0
					…				….			
$u_1=v_1$	0	0	0	0	…	0	1	0	0	0	…	0
$u_2=v_2$	0	0	0	0	…	0	0	1	0	0	…	0
					…				….			
W	1	1	1	1	…	1	3	3	3	3	…	3

Matching Problems

- **Perfect Bipartite Matching**
  - Given a bipartite graph $G=(V,E)$ where $V=X\cup Y$ and $E \subseteq X \times Y$, is there a set M in E such that every vertex in V is in precisely one edge of M ?

- In **P**
  - Network Flow gives $O(nm)$ algorithm where $n=|V|$, $m=|E|$.

3-Dimensional Matching

- **Perfect Bipartite Matching** is in **P**
  - Given a bipartite graph $G=(V,E)$ where $V=X \cup Y$ and $E \subseteq X \times Y$, is there a subset **M** in **E** such that every vertex in **V** is in precisely one edge of **M** ?

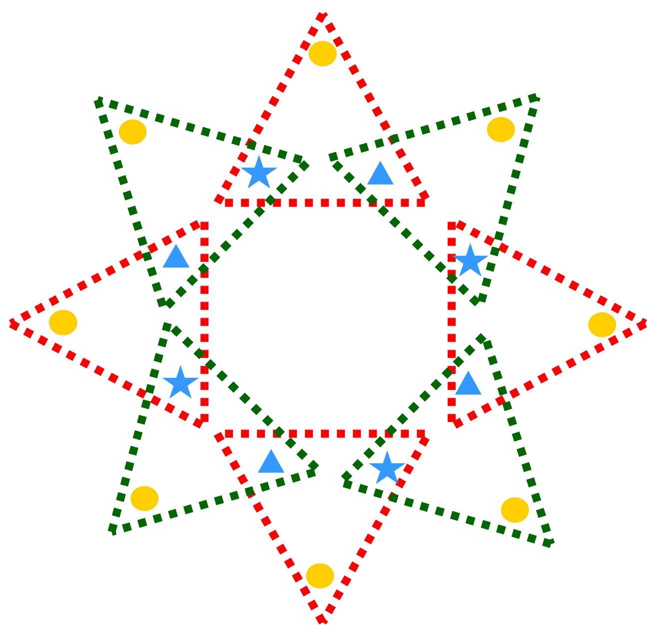
- **3-Dimensional Matching**
  - Given a tripartite hypergraph $G=(V,E)$ where $V=X \cup Y \cup Z$ and $E \subseteq X \times Y \times Z$, is there a subset **M** in **E** such that every vertex in **V** is in precisely one hyperedge of **M** ?
    - is in **NP:** Certificate is the set **M**

3-Dimensional Matching

- Theorem: **3-Dimensional Matching** is **NP**-complete
- Proof:
  - We've already seen that it is in **NP**
  - **3-Dimensional Matching** is **NP**-hard:
    - Reduction from **3-SAT**
    - Given a 3-CNF formula **F** we create a tripartite hypergraph ("hyperedges" are triangles) **G** based on **F** as follows

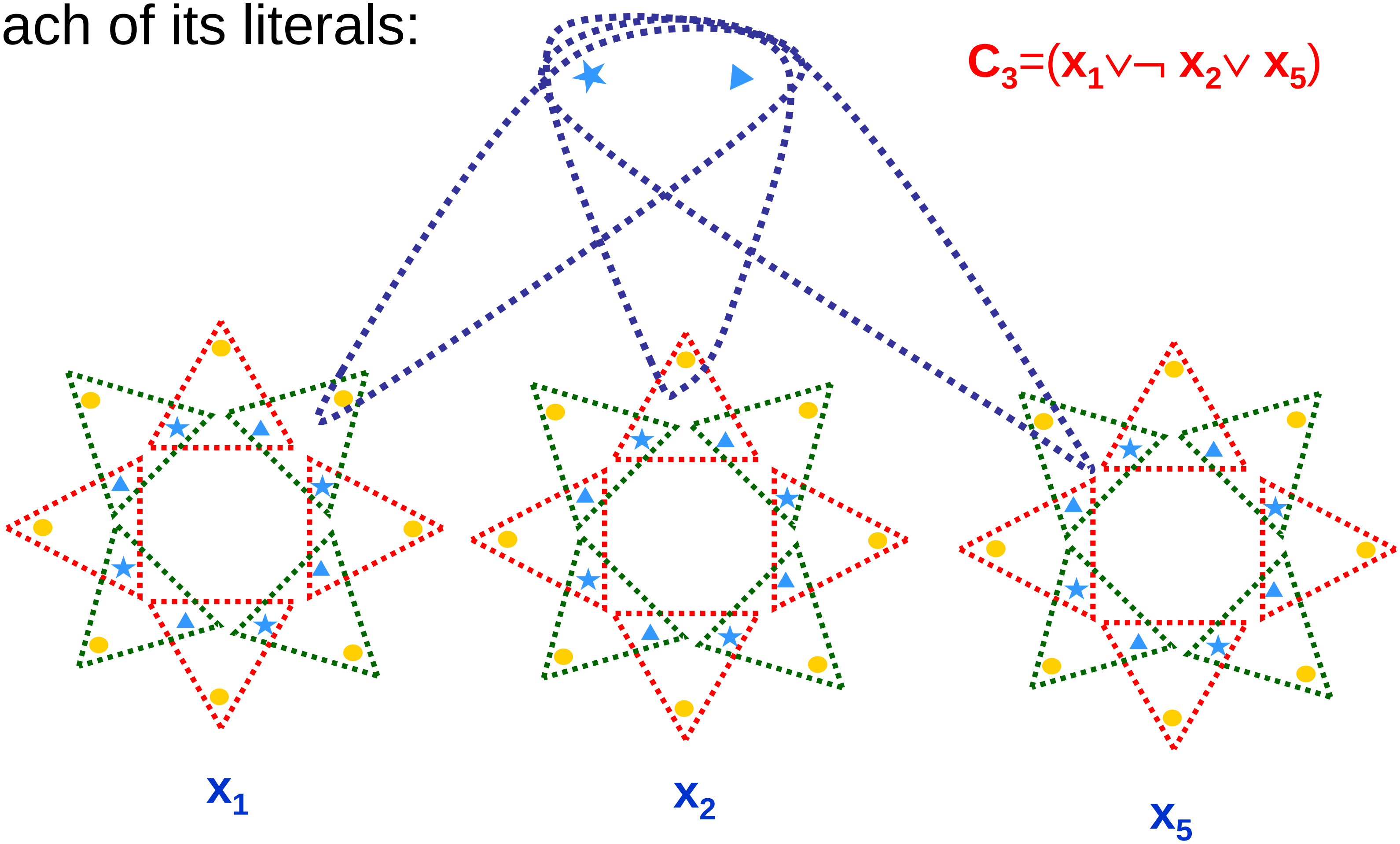
3-SAT $\leq_P$ 3-Dimensional Matching

- Variable part:
  - If variable x_i occurs r_i times in **F** create r_i red and r_i green triangles linked in a circle, one pair per occurrence
    - Perfect matching **M** must either use all the green edges leaving red tips uncovered (x_i is assigned false) or all the red edges leaving all green tips uncovered (x_i is assigned true)



3-SAT $\leq_P$ 3-Dimensional Matching

- Clause part: Two new nodes per clause joined to each of its literals:

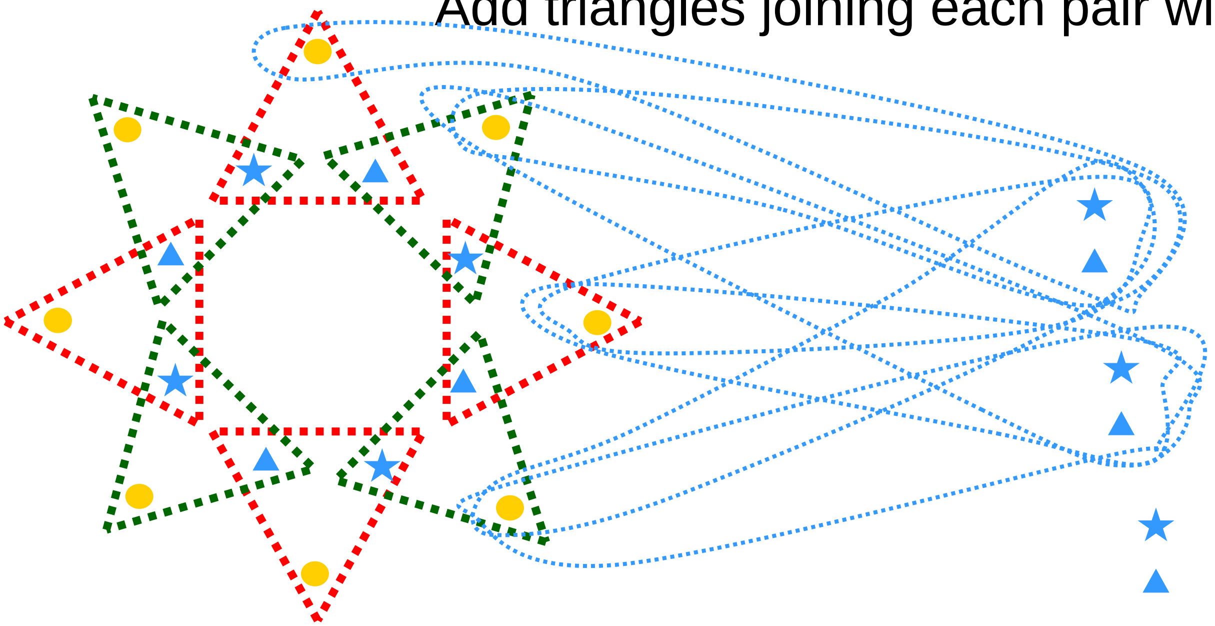


3-SAT $\leq_P$ 3-Dimensional Matching

- Slack: If there are **m** clauses then there are **3m** variable occurrences. That means **3m** total tips are not covered by whichever of red or green triangles not chosen. Of these, **m** are covered if each clause is satisfied. Need to cover the remaining **2m** tips.

Solution: Add **2m** pairs of slack vertices

Add triangles joining each pair with every tip!



3-SAT $\leq_P$ 3-Dimensional Matching

- Well-formed: Each triangle has one of each type of node: ★ ▲ ●
- Correctness:
  - If **F** has a satisfying assignment then choose the following triangles which form a perfect 3-dimensional matching in **G**:
    - Either the red or the green triangles in the cycle for $\mathbf{x_i}$ - the opposite of the assignment to $\mathbf{x_i}$
    - The triangle containing the first true literal for each clause and the two clause nodes
    - **2m** slack triangles one per new pair of nodes to cover all the remaining tips

3-SAT $\leq_P$ 3-Dimensional Matching

- Correctness continued:
  - If **G** has a perfect 3-dimensional matching then:
    - Each blue node in the cycle for each x_i is contained in exactly two triangles, exactly one of which much be in **M**. If one triangle in the cycle is in **M**, the others must be the same color. We use the color not used to define the truth assignment to x_i
    - The two nodes for any clause must be contained in an edge which must also contain a third node that corresponds to a literal made true by the truth assignment. Therefore the truth assignment satisfies **F** so it is satisfiable.