CSE521 Homework 4

Reading: Lecture notes and KT Chapter 11

Problem 1. A **legal k-coloring** of a graph is an assignment of colors $1, 2, \ldots, k$ to the vertices of the graph such that no two adjacent vertices receive the same color. A graph is **k-colorable** if there exists a legal k-coloring of its vertices. The problem of finding a legal k-coloring of a k-colorable graph is NP-complete for $k \geq 3$.

1. Prove that graphs with maximum degree Δ are $(\Delta+1)$-colorable. Also give a polynomial-time algorithm for finding a $(\Delta + 1)$-coloring.

2. Give a polynomial-time algorithm for 2-coloring a bipartite graph.

3. Using parts (a) and (b) above, give a polynomial-time algorithm for finding an $O(\sqrt{n})$-coloring of a 3-colorable graph.

   Hint: Verify and use the fact that the neighborhood of any vertex in a 3-colorable graph is 2-colorable.

Problem 2. Consider the following variant of the set cover problem. We are given a universe U of n elements and a collection $\mathcal{F} = \{S_1, S_2, \ldots, S_m\}$ of subsets of U. The goal is to pick a subfamily $\mathcal{G}$ of $\mathcal{F}$ to maximize the number of elements of U which are covered *exactly once* by this subfamily.

1. Suppose each element of U is present in exactly k sets. Give a randomized algorithm that outputs a subfamily which uniquely covers a number of elements which is in expectation at least $1/e$ times the optimal value.

   How does your analysis change if each element u is contained in k_u of the sets, where $k \leq k_u \leq 2k$ for all $u \in U$?

2. Using the above algorithm and classifying elements into suitable groups, obtain an $O(\log B)$-approximation algorithm for the general problem, where B is the maximum number of sets to which any element of U belongs.

Problem 3. Show that the following **Quadratic Programming** problem is NP-hard. You are given a set of equations $E_1, E_2, \ldots, E_t$ in n variables $x_1, x_2, \ldots, x_n$, where E_k is of the form $\sum_i a_{i,k} x_i^2 + \sum_{i,j} b_{i,j,k} x_i x_j + \sum_i d_{i,k} x_i = c_k$ where $a_{i,k}, b_{i,j,k}, d_{i,k}, c_k$ are given integers. The problem is to decide if there is an assignment of real numbers $v_1, \ldots, v_n$ to the variables $x_1, \ldots, x_n$ which makes all the equations simultaneously true.