**Def** Factorization (F)

$P(x)>0$ factorizes accd to G if $\exists f_c$

$$P(x) = \frac{1}{Z} \prod_{c \in \mathcal{C}} f_c(x_c)$$

Norm. term

Maximal Cliques

---

Does (G) $\Rightarrow$ (F) ?

$\boxed{\text{Thm}}$ [Hammersley - Clifford Thm]

When $P(X) > 0 \quad \forall \ x \in \mathcal{X}^n$,

(G) $\Rightarrow$ (F).

---

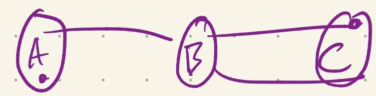
Implications when $P(X)>0 \ \forall x \in \mathcal{X}^n$

(F) $\not\Leftarrow\Rightarrow$ (G) $\Leftarrow\Rightarrow$ (L) $\Leftarrow\Rightarrow$ (P)

---

Claim (F) $\Rightarrow$ (G)

Proof: Fix any A - B - C in G




$\nexists \ a \in A, \ c \in C$ s.t. $(a, c) \in E$

$\forall a \in A, \ c \in C \quad (a, c) \notin E$

$\downarrow$

all cliques contain nodes in A, or C, or neither, but not both!

$$P(X) = \frac{1}{Z} \prod_{c \in \mathcal{C}} f_c(x_c)$$

$$= \frac{1}{Z} \underline{F_1}(x_A, x_B) \cdot F_2(x_B, x_c)$$

(HW1) $\Rightarrow$

$X_A \perp\!\!\!\perp X_C \mid X_B$.

$\square$

**Proof of HCT** Assume $P(x) > 0 \quad \forall x \in \mathcal{X}^n$

$X_A \perp\!\!\!\perp X_C \mid X_B \quad \forall\ A - B - C$ sep in $G$

WTS : $P(x) \propto \prod_{c \in \mathcal{C}} f_c(x_c)$ [e.g. construct factors $f_c$]

$\boxed{\text{Def}}$ Pseudofactors

$$\tilde{f}_S(x_S) = \prod_{U \subseteq S} P[\underbrace{X_U, \vec{O}_{rest}}]^{(-1)^{|S \setminus U|}}$$

$\mathcal{X}^n \to [0,1]$

Pick arbitrary $0 \in \mathcal{X}$

Well-defined? [Not / by 0 anywhere?]
B/C $P(x) > 0 \quad \forall\ x \in \mathcal{X}^n$

$\boxed{\text{Claim 1}}$ When $S$ is not a clique, $\tilde{f}_S(x_S)$ is a constant $\left[\begin{matrix} P(x)>0 \\ (6) \end{matrix}\right]$

$\boxed{\text{Claim 2}}$ $P(x) = P(0) \cdot \prod_{S \subseteq V} \tilde{f}_S(x_S)$ $[P[x] > 0,\ \text{Claim 1}]$

$$\Rightarrow P(x) = P(0) \prod_{c \in \mathcal{C}} \tilde{f}_c(x_c)$$

$\square$

# Pseudofactor examples

Singletons $\qquad \tilde{f}_{\{i\}}(X_i) = \dfrac{P(X_i, \vec{O}_{rest})}{P(O_i, \vec{O}_{rest})}$ $\qquad$ Subsets: $\emptyset$, $\{i\}$

Pairwise $\qquad \underset{S}{\underline{\tilde{f}_{\{i,j\}}}}(X_i, X_j) = \dfrac{P(X_i, X_j, \vec{O}_{rest}) \cdot P(O_i, O_j, \vec{O}_{rest})}{P(X_i, O_j, \vec{O}_{rest}) \cdot P(X_j, O_i, \vec{O}_{rest})}$ $\qquad \emptyset$, $\{i\}$, $\{j\}$, $\{i,j\}$

---

$\boxed{\text{Claim 1}}$ When $S$ is not a Clique, $\tilde{f}_S(X_S)$ is a constant $\left[\begin{matrix} P(X)>0 \\ (G) \end{matrix}\right]$

$\boxed{\text{Claim 2}}$ $P(X) = P(O) \cdot \prod_{S \subseteq V} \tilde{f}_S(X_S)$ $\qquad$ [$P[X] > 0$, Claim 1]

## Proof of Claim 1

we know

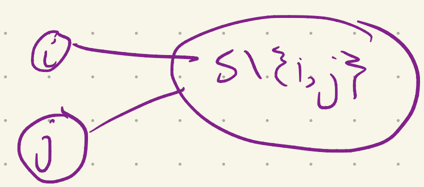
$$\tilde{f}_S(X_S) \triangleq \prod_{U \subseteq S} P[X_U, \vec{O}_{rest}]^{(-1)^{|S \setminus U|}},$$

$$= \prod_{U \subseteq T} \left[ \frac{P[X_U, X_i, X_j, \vec{O}_{rest}]\, P[X_U, O_i, O_j, \vec{O}_{rest}]}{P[X_U, X_i, O_j, \vec{O}_{rest}]\, P[X_U, X_j, O_i, \vec{O}_{rest}]} \right]^{(-1)^{|T \setminus U|}}$$

$S$ is not a clique

$\Rightarrow \exists\, i, j \in S$

$(i, j) \notin E$



$$\frac{P[x_u, x_i, x_j, \vec{O}_{rest}]}{P[X_u, x_i, O_j, \vec{O}_{rest}]} = \frac{\cancel{P[x_i \mid x_u, O_{rest}, X_j]} \cdot P[x_u, X_j, \vec{O}_{rest}]}{\cancel{Pr[x_i \mid X_u, O_{rest}, O_j]} \cdot P[X_u, O_j, \vec{O}_{rest}]} = \frac{A}{B}$$

B/C $x_i \perp\!\!\!\perp x_j \mid x_u, O_{rest}$

$(G) \Rightarrow (P)$

$$\frac{P[X_u, O_i, O_j, \vec{O}_{rest}]}{P[X_u, X_j, O_i, \vec{O}_{rest}]} = \frac{Pr[O_i \mid \cancel{X_u, O_j, \vec{O}_{rest}}] \cdot Pr[X_u, O_j, \vec{O}_{rest}]}{Pr[O_i \mid \cancel{X_u, X_j, \vec{O}_{rest}}] \cdot Pr[X_u, X_j, \vec{O}_{rest}]} = \frac{C}{D}$$

$$\tilde{f}_S(x_S) = \prod_{U \subseteq T} \left( \frac{A \cdot D}{B \cdot C} \right)^{(-1)^{|T \setminus S|}}$$

$$= 1, \text{ a constant!} \qquad \square$$

$\boxed{\text{Claim 2}}$ Write-up.

Notes : MRF's $(G) \Rightarrow (L) \Rightarrow (P)$ $\quad$ $P(x)>0$

(F) Factorization [$P(X)$ over $G$ factorizes]

$F \Rightarrow G$

$P(X)>0 \quad G \Rightarrow F$

Lecture up tomorrow (Hammerseley-Clifford thm)

---

**Claim:** $\exists\, P(X)$ s.t. $\nexists\, G$ (directed or undirected) where $I(P) = I(G)$

**Def** $G$ an I-map of $P$ if $I(G) \subseteq I(P)$

P-map if $I(G) = I(P)$

**Def:** $G$ is a minimal I-map if you can't remove edges + remain an I-map.

---

How do we construct I-MAPs?

- For BNs
  - Fix an ordering $\{1, \dots, n\}$
  - Remove edges in turn if doing so maintains $I(G') \subseteq I(P)$
- Not unique (ordering matters)

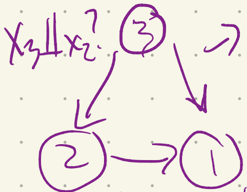
  #edges isn't even predetermined.

Ex

$P(X) \quad I(P) = \{X_1 \perp\!\!\!\perp X_2\}$

(1) ordering 3, 2, 1

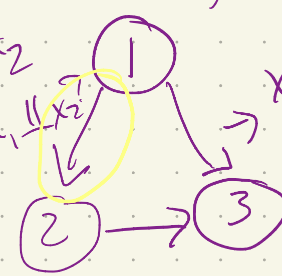
$X_3 \perp\!\!\!\perp X_2$? $\quad \to X_1 \perp\!\!\!\perp X_3 | X_2$



$\to X_1 \perp\!\!\!\perp X_2 | X_3$

(2) Ordering 1, 2, 3

$X_1 \perp\!\!\!\perp X_2$? $\quad \to X_3 \perp\!\!\!\perp X_1 | X$

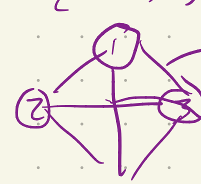


---

- For MRFs, Remove any edge that can be removed

  Claim: If $P(X)>0$, minimal I-MAP is unique

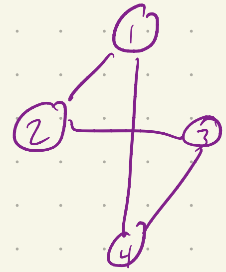
$V = \{1, 2, 3, 4\} \quad I(P) = \{X_1 \perp\!\!\!\perp X_3 | X_2, X_4 \quad X_2 \perp\!\!\!\perp X_4 | X_1, X_3$



$\to X_1 \perp\!\!\!\perp X_3 | X_2, X_4$? Yes

$\to X_2 \perp\!\!\!\perp X_3 | X_1, X_4$ X

When can we convert between MRFs + BNs?



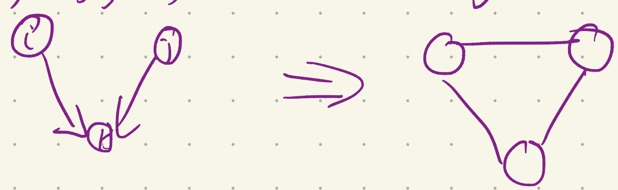

? DAG $\Rightarrow$ Undirected

$\boxed{\text{Def}}$ [Moralization]

Fix a DAG $G=(V,E)$. A moralization of $G$ is an undirected $G'=(V,E')$

where $(i,j)\in E \Rightarrow (i,j)\in E'$

$(i,k)\ (j,k)\in E \Rightarrow (i,j)\in E'$




Claim

$I(\text{moral } G') \subseteq I(\text{DAG } G)$

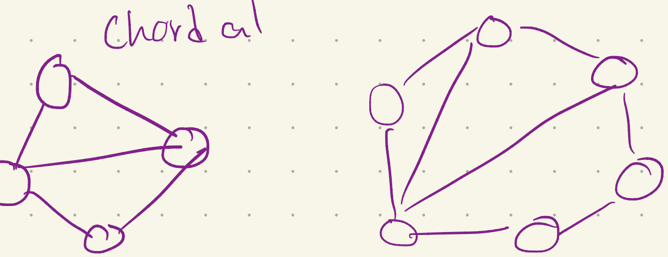
If moral $G'$ doesn't add edges,

$I(G)=I(G')$

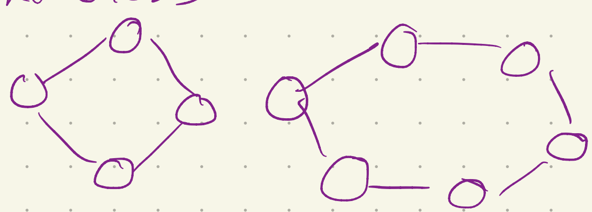
? Undirected $G \Rightarrow$ DAG $G$

Def [Chordal graph] $G=(V,E)$ an undirected graph is chordal if every loop of size $\geq 4$ has a chord (an edge between some nonadj nodes $\in$ loop

Chordal



Chordless



Claim Undirected $G$ is chordal $\Leftrightarrow \exists$ a DAG $G'$

$I(G)=I(G')$

Set of all I-MAPs for P