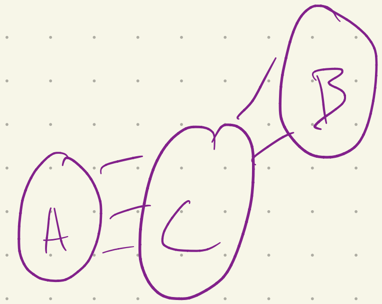
Undirected GMs aka MRFs

Graph factorizes wrt G as $P(x) = \frac{1}{Z} \prod_{c \in \mathcal{C}} f_c(x_c)$

$$Z \triangleq \sum_{x \in \mathcal{X}^n} \left(\prod f_c(x_c) \right)$$

Graph separation (GS) A & C are separated by B "A – B – C"



Global Markov (G) w.r.t. G

P(x) satisfies GM wrt G iff $\forall$ A – B – C

$$X_A \perp\!\!\!\perp X_C \mid X_B$$

[Factorization according to G $\Rightarrow$ GM wrt G]

Local Markov Property (L)

P(x) satisfies L wrt G if

$$X_i \perp\!\!\!\perp X_{V \setminus (i \cup \delta(i))} \mid \delta(i)$$

Neighbors of i

Pair Wise Property (P)

$P(x)$ satifies (P) wrt G if

$$X_i \perp\!\!\!\perp X_j \mid X_{V \setminus \{i,j\}} \quad \forall (i,j) \notin E$$

Claim | $\underbrace{G \Rightarrow L}_{①} \underbrace{\Rightarrow P}_{②}$

Lemma [Data Processing #]

- $X - Z - Y - h(Y)$

(1) $X \perp\!\!\!\perp Y \mid Z \Rightarrow X \perp\!\!\!\perp h(Y) \mid Z$

(2) $X \perp\!\!\!\perp Y \mid Z, h(Y)$

Proof that $G \Rightarrow L$

We know $\forall A, B, C$ s.t. $A - B - C$ in G,

$$X_A \perp\!\!\!\perp X_C \mid X_B$$

In Particular $A = \{i\}$ $\quad B = \{\delta(i)\}$ $\quad C = V \setminus (A \cup B)$

then $A - B - C$ since $(i,j) \in E \Rightarrow j \in B$

$\Rightarrow a - b_i \cdots c \Rightarrow X_i \perp\!\!\!\perp X_{V \setminus \_} \mid X_{\delta_i}$

Proof that L $\Rightarrow$ P

We know $X_i \perp\!\!\!\perp X_{V\setminus\{i\cup\delta_i\}} \mid X_{\delta_i} \;\forall i.$

P $\forall j: (i,j) \notin E \Rightarrow j \notin \delta(i)$

(DP2)(a) $\Longrightarrow X_i \perp\!\!\!\perp X_{V\setminus(i\cup\delta_i)} \mid X_{V\setminus\{i,j\}}$

(DP1)(b) $\Rightarrow X_i \perp\!\!\!\perp X_j \mid X_{V\setminus\{i,j\}}$ $j \in$

which is (P)

$\square$

Assume our PMF P has $P(X) = P(X_1, \ldots X_n) > 0$

$\forall\ X = [X_1, \ldots, X_n] \in \mathcal{X}^n$

Claim: If $P(x) > 0 \;\; \forall x \in \mathcal{X}^n$ then $\underset{\text{pairwise}}{(P)} \Rightarrow \underset{\text{global}}{(G)}$

$(G) \Rightarrow (L) \Rightarrow (P)$

$\{$if $P(x) > 0\}$

$\boxed{\text{Lemma}}$: Intersection Lemma

Fix $P(x) > 0$.

suppose
- $X_A \perp\!\!\!\perp X_B \mid (X_C, X_D)$

and
- $X_A \perp\!\!\!\perp X_C \mid (X_B, X_D)$

then $X_A \perp\!\!\!\perp X_B, X_C \mid X_D$

Proof of Claim [$P(x) > 0 \;\; \forall x$ implies $(P) \Rightarrow (G)$]

$(P) \;\; \star \;\; \forall (i,j) \notin E, \;\; X_i \perp\!\!\!\perp X_j \mid X_{V \setminus \{i,j\}}$

want to show $\star \Rightarrow X_A \perp\!\!\!\perp X_C \mid X_B \;\; \forall \; A - B - C$

Will show by induction on $|B| = n-2, n-3, \dots, \underline{1}$

(1) $|B| = n-2$, $A = \{i\}$ $C = \{j\}$. Since $(i,j) \notin E$,

$$\{i\} - V\setminus\{i,j\} - \{j\}$$

and so $X_A \perp\!\!\!\perp X_C \mid X_B$

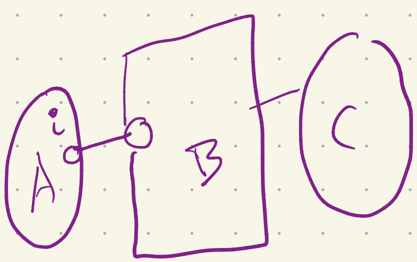
(2) Assume (G) holds $\forall\ |B| \geq s$.

$|B| = s-1 < n-2$

Assume wlog $|A| \geq 2$ & $A - B - C$ are separated in G

Take $i \in A$ adjacent to B. Then,

$$\underbrace{A\setminus\{i\}}_{\tilde{A}} - \underbrace{B \cup \{i\}}_{\tilde{B}} - C \quad \text{also separated}$$



$\tilde{B}$, $|\tilde{B}| = s$ $\overset{IH}{\Longrightarrow}$ (1) $X_C \perp\!\!\!\perp X_{A\setminus\{i\}} \mid X_{B\cup\{i\}}$

$$\{i\} - \underbrace{B \cup (A\setminus\{i\})}_{\text{size} \geq s} - C \overset{IH}{\Longrightarrow} (2)\ X_C \perp\!\!\!\perp X_i \mid X_B, X_{A\setminus\{i\}}$$

Intersection Lemma + (1, 2) $\Longrightarrow$ $X_C \perp\!\!\!\perp X_A \mid X_B$ $\forall$ $A - B - C$ sep + $|B| = s$. ∎

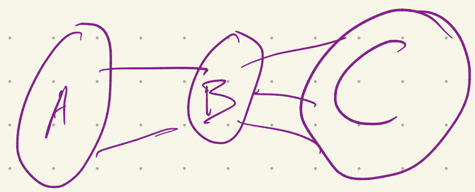
The intersection lemma uses $P(X)>0$, and we actually need it or $\exists$ conterexample $G \not\Rightarrow P$ [HW5]

Def Factorization (F): We say $P(x)$ factorizes wrt G if

$$P(x) = \frac{1}{Z} \sum_{c \in \mathcal{C}} f_c(x_c)$$

Maximal clique set

$\boxed{\text{Claim: } (F) \Rightarrow (G)}$



Proof: We have ① $P(x) \propto \prod_{c \in \mathcal{C}} f_c(x_c)$ & ② $A - B - C$

want to show that $X_A \perp\!\!\!\perp X_C \mid X_B$

$\propto F_{AB}(x_A, x_B)\, F_{BC}(x_B, x_C)$

No factor has both $a \in A$, $c \in C$. □

$X_A \perp\!\!\!\perp X_C \mid X_B$

Claim $(G) \Rightarrow (F)$ if $P(X) > 0$ (HW1)

Hammerseley-Clifford thm

So, if $P(X)>0$

$(F) \Leftrightarrow (G) \Leftrightarrow L \Leftrightarrow (P)$

$\boxed{\text{Not true}}$ $G \not\Rightarrow F$ when $P(x)>0$

Proof G ⇒ F

Fix (P, G) s.t. $P(x) > 0$, P satisfies (G)lobal Markov wrt G. $(X_A \perp\!\!\!\perp X_C \mid X_B \;\; \forall \; A - B - C \in G)$

Want to find factorization s.t. $P(x) \propto \prod_{c \in \mathcal{C}} f_c(x_c)$

Def "Pseudo-factors"

$$\tilde{f}_S(x_S) \triangleq \prod_{U \subseteq S} P(\overbrace{X_U, 0_{V \setminus U}}^{x^n})^{(-1)^{|S \setminus U|}}$$

$\hookrightarrow X_{S \setminus U} = \vec{0}$

$0 \in \mathcal{X}$

Since $P(X) > 0$

Example: the singleton pseudofactors:

$\binom{n}{1}$ singletons $\tilde{f}_i(x_i) = \dfrac{P(x_i, 0^{n-1})}{P(X_i = 0, X_{-i} = 0^{n-1})}$

Pairwise $f_{ij}(X_{ij}) = \dfrac{P(x_i, x_j, 0^{n-2}) \cdot P(x_i = x_j = 0, 0^{n-2})}{P(x_i, x_j = 0, 0^{n-2}) \, P(x_i = 0, x_j, 0^{n-2})}$

Recall

[Bayes Nets / Directed GMs]

$I(G)$ = The set of all $\perp\!\!\!\perp$ implied by $G=(V,E)$

$I(P)$ = the set of all $\perp\!\!\!\perp$ implied by factorization of P

- $\exists\, P(x)$ where no $\nexists\, G$ with $I(G) = I(P)$

$$X_1 = X_2$$

$$X_3 = X_1 + Z = X_2 + Z$$

$$P(X_1, X_2, X_3)$$

$$= \begin{cases} 0 & \text{if } X_1 \neq X_2 \\ P(X_1)\,P(Z = X_3 - X_1 \mid X_1) \end{cases}$$

$$\underline{P(X_1, X_2)\,P(X_3 \mid X_1, X_2)}$$

- $\forall G, \exists P$ s.t. $I(G) = I(P)$
- $\exists G_1 \neq G_2$ s.t. $I(G_1) = I(G_2)$

FIX $\circ \to \circ \to \circ$ X_1, $X_2 | X_1$, $X_3 | X_2$ $\circ \leftarrow \circ \to \circ$ X_1, $X_2 | X_1$, $X_3 | X_1$

find example

Def G is a perfect map for P if $I(G) = I(P)$

G is an I-map for P if $I(G) \subseteq I(P)$

G is a minimal I-map for P if

- It is an I-Map for P
- $G' = (V, E \setminus (i,j))$ isn't an imap $\forall (i,j) \in E$

4 Minimal IMap isn't unique

- ordering of variables
- Even fixing ordering

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] \arrow[rr, bend left] & 2 & 3 \end{tikzcd}$$

$X_1 = X_2$

$X_3 = X_1 + Z$

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5 $I(G) = \bigcap I(P)$ where P factorizes wrt G